with $4p^2 > \mu^2$. Find the angular displacement $\theta(t)$, and the time t and angular displacement when it first comes to rest, given that it starts with $\theta = 0$ and $d\theta/dt = \alpha$.

42. The top of a vibration damper oscillates in a straight line in such a way that its position x from the origin at time t obeys the differential equation $x'' + 2x' + 4x = 0$. Given that it starts from the origin with speed U, find its position as a function of U and t and the distance from the origin when it first comes to rest.

43. The free oscillations of all physical systems giving rise to oscillatory solutions obey an equation of the form $x'' + 2\mu x' + (\mu^2 + p^2)x = 0$ with $p^2 > 0$. Given that $x(0) = 0$, and $(dx/dt)_{t=0} = Ap$, solve the equation and show that $x(t) = A\exp(-\mu t)\sin pt$. Use this result to prove that the ratio of the magnitude of successive extrema of $x(t)$ forms a geometric series with common ratio $r = \exp[-\mu\pi/(p)]$. The number $\mu\pi/(p)$ is called the **logarithmic decrement** of the oscillations.

6.2 Oscillatory Solutions

The nonhomogeneous constant coefficient second order equation

$$a_0\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_2y = f(t), \tag{21}$$

in which t can be regarded as the time and $f(t)$ as an external input to the system, is the simplest mathematical model capable of representing the oscillatory behavior of a physical system.

It was shown in Section 5.2(d) that one way this equation can arise is when describing the motion of a mass–spring system in which a mass moves on a rough horizontal surface, with the motion resisted by a frictional force proportional to the speed. Friction dissipates energy, so the motion will decay to zero as time increases unless it is sustained by some external input of energy in the form of a **forcing function** represented in (21) by the nonhomogeneous term $f(t)$. The dissipation of energy due to friction, or to a friction-like effect in other applications, is called **damping**, and in the R–L–C circuit considered in Section 5.2(d), where the charge q on the capacitor was shown to satisfy a homogeneous form of equation (21) with $a_0 = LC$, $a_1 = RC$, and $a_3 = 1$, the damping was due to the dissipative (friction-like) term $a_1 = RC$.

forcing function and damping

Another way in which equation (21) can arise is when a cylindrical mass with moment of inertia I about its axis of symmetry is mounted on a flexible shaft that can be twisted about its axis, with the resistance to **torsion** (twisting) proportional to the angle of twist θ, and damping proportional to the angular velocity $d\theta/dt$ about the shaft. This occurs, for example, in a torsional pendulum and also in heavy rotating machinery when a heavy flywheel is attached to a shaft. The equation governing the **torsional oscillations** $\theta(t)$ as a function of the time t becomes

$$I\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + \mu\theta = f(t),$$

where k and μ are constants and, as before, $f(t)$ is a forcing function.

A comparison of the second order constant coefficient differential equations that govern mechanical, electrical, and torsional oscillations leads to Table 6.1, which relates analogous physical quantities in each of the different systems.

Many other physical situations can be represented by this same constant coefficient second order differential equation with varying degrees of approximation. It does, for example, provide a simple model that describes the effect of a fluctuating vertical lift at the center of a flexible suspension bridge caused by gusts of wind. If

TABLE 6.1 A Comparison of Second Order Constant Coefficient Differential Equations

Mechanical System	Electrical System with Elements in Series	Torsional System
Mass	Inductance	Moment of inertia
Damping constant	Resistance	Torsional damping constant
Spring constant	Reciprocal of capacitance	Shaft torsional constant
Applied force	Applied voltage	Applied torque

this effect is sustained, and the gusts come at the natural frequency of the bridge, the amplitude of the oscillations can become dangerously large. On November 7, 1940, in the state of Washington, this effect caused the failure of the Tacoma Narrows Bridge over Puget Sound. Powerful gusting winds at around the natural frequency of this excessively flexible bridge induced and then sustained vertical oscillations of the bridge that reached an amplitude of 28 feet before the bridge snapped and fell.

When analyzing the oscillatory nature of solutions of (21), and looking at the effect of *resonance* that occurs if the natural frequency of oscillation of the system coincides with the frequency of a periodic forcing function, it is helpful to have in mind a mathematical model of a simple but typical mechanical system. The mechanical system we will consider here is shown in Fig. 6.2, and it involves a piece of heavy machinery that vibrates vertically and is mounted on a spring and damper system to reduce the transmission of the vibrations to the foundations of the building. The damper is usually a piston that moves in a viscous fluid, with the resisting force considered to be proportional to the speed of the piston.

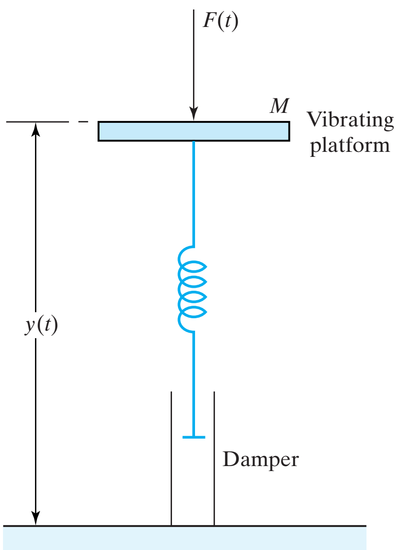


FIGURE 6.2 A vibrating machine mounted on a spring and damper system.

If the mass of the machine is M, the displacement of the machine from the floor at time t is $y(t)$, the spring constant is k, the constant of proportionality for the damper is μ, and the force exerted by the vibrating machine is $\tilde{F}(t)$, the rate of change of momentum $d/dt\{M(dy/dt)\}$ must be equated to the frictional resistance $-\mu dy/dt$, the restoring force of the spring $-ky$, and the external force $\tilde{F}(t)$. So this system, with the displacement $y(t)$ as its **one degree of freedom**, is seen to satisfy the differential equation

$$M\frac{d^2y}{dt^2} = -\mu\frac{dy}{dt} - ky + \tilde{F}(t).$$

For convenience this will be written in the standard form

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = F(t), \tag{22}$$

with $a = \mu/M$, $b = k/M$, and $F(t) = \tilde{F}(t)/M$.

Differential equation (22) is nonhomogeneous, so its solution will be more complicated than the solution of the homogeneous equation considered in the previous section. The equation is linear, so as in Section 5.6 we will represent its general solution as the sum

$$y(t) = y_c(t) + y_p(t), \tag{23}$$

with $y_c(t)$ the general solution of the homogeneous form of equation (22)

$$\frac{d^2y_c}{dt^2} + a\frac{dy_c}{dt} + by_c = 0, \tag{24}$$

and $y_p(t)$ a particular solution of

$$\frac{d^2y_p}{dt^2} + a\frac{dy_p}{dt} + by_p = F(t) \tag{25}$$

that contains no arbitrary constants.

The justification for writing the general solution of (22) as $y(t) = y_c(t) + y_p(t)$ follows if we notice that (22) can be written

$$\frac{d^2[y_c + y_p]}{dt^2} + a\frac{d[y_c + y_p]}{dt} + b[y_c + y_p] = F(t)$$

or, equivalently, as

$$\left[\frac{d^2y_c}{dt^2} + a\frac{dy_c}{dt} + by_c\right] + \frac{d^2y_p}{dt^2} + a\frac{dy_p}{dt} + by_p = F(t),$$

where the group of terms in the bracket vanishes because of (24), while $y_p(t)$ satisfies the remainder of the equation because of (25).

complementary function and particular integral

As in Section 5.6, the solution $y_c(t)$ will be called the **complementary function**, and the solution $y_p(t)$ will be called a **particular integral**. It is important to recognize that the two arbitrary constants associated with the general solution of (22) occur in the complementary function $y_c(t)$, whereas the particular integral $y_p(t)$ contains no arbitrary constants.

We now introduce the notation $a = 2\zeta$ and $b = \Omega^2$, when the characteristic equation of (22) becomes

$$\lambda^2 + 2\zeta\lambda + \Omega^2 = 0, \tag{26}$$

with the roots

$$\lambda_1 = -\zeta + (\zeta^2 - \Omega^2)^{1/2} \quad \text{and} \quad \lambda_2 = -\zeta - (\zeta^2 - \Omega^2)^{1/2}. \tag{27}$$

The solution $y_c(t)$ of (22) will correspond to one of the Cases (I), (II), or (III) of Section 6.1, but before determining its form in each of these cases we further simplify the notation by setting $k^2 = \zeta^2 - \Omega^2$, so that

$$\lambda_1 = -\zeta + k \quad \text{and} \quad \lambda_2 = -\zeta - k. \tag{28}$$

***Case (I):* $k^2 > 0\,(\zeta^2 > \Omega^2)$**

The complementary function $y_c(t)$ is nonoscillatory and given by

$$y_c(t) = \exp(-\zeta t)\{C_1 \exp(kt) + C_2 \exp(-kt)\}. \tag{29}$$

***Case (II):* $k^2 < 0\,(\zeta^2 < \Omega^2)$**

If we set $k^2 = -\omega_0^2$ the complementary function is seen to be oscillatory and given by

$$y_c(t) = \exp(-\zeta t)\{C_1 \cos\omega_0 t + C_2 \sin\omega_0 t\}. \tag{30}$$

***Case (III):* $k^2 = 0\,(\Omega^2 = \zeta^2)$**

The complementary function is nonoscillatory and given by

$$y_c(t) = \{C_1 + C_2 t\}\exp(-\zeta t). \tag{31}$$

critical damping and overdamping

transient and steady state solutions

Cases (I) and (III) exhibit no oscillatory behavior. Case (I) is said to be **overdamped** and Case (III) to be **critically damped**, because it marks the boundary between the overdamped behavior of Case (I) and the oscillatory behavior of Case (II). The parameter ζ entering into the exponential factor $\exp(-\zeta t)$ that is present in all three cases is called the **damping exponent** and, provided $\zeta > 0$, the factor $\exp(-\zeta t)$ will cause all three complementary functions to decay to zero as time increases. This property of a complementary function with $\zeta > 0$ has led to its being called the **transient solution** of the differential equation. Accordingly, after a suitable lapse of time, only the particular integral $y_p(t)$ will remain, and this property is recognized by calling $y_p(t)$ the **steady state solution**, with the understanding that it is the *time-dependent* solution that remains after the transient solution has become vanishingly small.

Typical transient solution behavior in the critically damped case is shown in Fig. 6.3 for different initial conditions, some of which can cause an initial increase in the amplitude of $y_c(t)$ before it decays to zero. The behavior in the overdamped case is similar to that in the critically damped case.

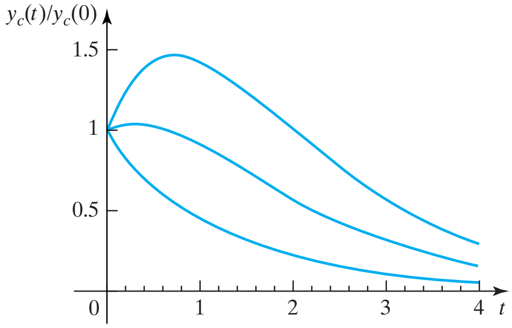


FIGURE 6.3 $y_c(t)$ in the critically damped case for different initial conditions.

It is now necessary to determine the form of the particular integral $y_p(t)$, and to do so the function $F(t)$ must be specified. A vibration is periodic in nature, so we shall model it by a nonhomogeneous term of the form

$$F(t) = A\cos\omega t, \tag{32}$$

amplitude and angular frequency of a vibration

in which the **amplitude** A will be considered to be fixed and the **angular frequency** ω will be regarded as a parameter. The angular frequency ω is expressed in terms of radians/unit time and corresponds to a time period of oscillation of $T = 2\pi/\omega$ time units, while the **frequency** of the vibration $1/T = \omega/2\pi$ measures the number of cycles (vibrations) occurring in one time unit. If the unit of time T is 1 sec, the frequency is measured in cycles/sec, called **hertz** (Hz), so 20 Hz is 20 cycles/sec.

Setting $F(t) = A\cos\omega t$ in (22) shows that the differential equation to be considered is

$$\frac{d^2y}{dt^2} + 2\zeta\frac{dy}{dt} + \Omega^2 y = A\cos\omega t. \tag{33}$$

A systematic approach to the determination of particular integrals will be developed in the next section, but here we will proceed from first principles. As equation (33) has constant coefficients, and its nonhomogeneous term is $A\cos\omega t$, the way this nonhomogeneous term can be obtained by differentiating a particular integral $y_p(t)$ is if the particular integral is of the form

$$y_p(t) = C\sin\omega t + D\cos\omega t, \tag{34}$$

unless $\zeta = 0$ and $\Omega = \omega$ (then try $y_p = t(C\sin\omega t + D\cos\omega t)$).
Substituting (34) into (33) and collecting terms gives

$$[(\Omega^2 - \omega^2)C - 2\zeta\omega D]\sin\omega t + [(\Omega^2 - \omega^2)D + 2\zeta\omega C]\cos\omega t = A\cos\omega t.$$

This must be true for all t, but this will only be possible if the respective coefficients of sin ωt and cos ωt on either side of the equation are identical, so

$$(\Omega^2 - \omega^2)C - 2\zeta\omega D = 0 \quad \text{and} \quad (\Omega^2 - \omega^2)D + 2\zeta\omega C = A.$$

Solving for C and D gives

$$C = \frac{2A\omega\zeta}{(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2} \quad \text{and} \quad D = \frac{A(\Omega^2 - \omega^2)}{(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2}, \tag{35}$$

so the required particular integral is

$$y_p(t) = \frac{2A\omega\zeta}{(\Omega^2-\omega^2)^2+4\zeta^2\omega^2}\sin\omega t + \frac{A(\Omega^2-\omega^2)}{(\Omega^2-\omega^2)^2+4\zeta^2\omega^2}\cos\omega t. \tag{36}$$

A better understanding of the nature of this particular integral can be obtained if it is rewritten. To accomplish this we return to (34) and write it as

$$y_p(t) = (C^2+D^2)^{1/2}\left[\frac{C}{(C^2+D^2)^{1/2}}\sin\omega t + \frac{D}{(C^2+D^2)^{1/2}}\cos\omega t\right], \tag{37}$$

and then define an angle ϕ by the requirement that

$$\sin\phi = \frac{C}{(C^2+D^2)^{1/2}}, \quad \text{and} \quad \cos\phi = \frac{D}{(C^2+D^2)^{1/2}}, \tag{38}$$

or by the equivalent expression

$$\tan\phi = C/D = \frac{2\zeta\omega}{\Omega^2-\omega^2}. \tag{39}$$

The trigonometric identity $\cos(\omega t-\phi) = \cos\omega t\cos\phi + \sin\omega t\sin\phi$ then allows $y_p(t)$ to be expressed in the more convenient form

$$y_p(t) = \frac{A}{[(\Omega^2-\omega^2)^2+4\zeta^2\omega^2]^{1/2}}\cos(\omega t-\phi). \tag{40}$$

Using this form for $y_p(t)$ gives the simpler expression for the general solution

$$y(t) = y_c(t) + \frac{A}{[(\Omega^2-\omega^2)+4\zeta^2\omega^2]^{1/2}}\sin(\omega t-\phi), \tag{41}$$

where $y_c(t)$ is one of the Cases (I), (II), or (III), depending on the sign of $\zeta^2-\Omega^2$.

phase angle and phase lag

The angle ϕ, which by convention is required to lie in the interval $0<\phi<\pi$, is called the **phase angle** of the solution, and often the **phase lag**, because it represents the *delay* with which the steady-state solution (the *output* from the system) lags behind the *input* to the system determined by $F(t)$.

We have seen that provided $\zeta>0$, the transient solution $y_c(t)$ decays to zero as t increases, leaving only the **steady state** solution $y_p(t)$. The steady state solution (41), illustrated in Fig. 6.4, is a sinusoid with the same angular frequency as the function $F(t) = A\sin(\omega t)$ that forces the oscillations, but with an amplitude that depends on both A and the angular forcing frequency ω. The effect of the phase lag ϕ is seen to shift the origin.

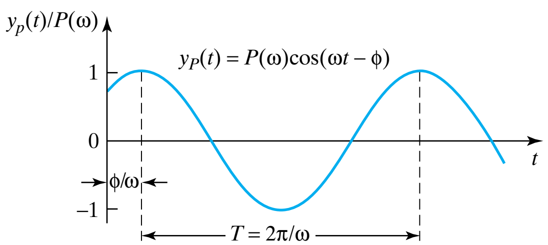


FIGURE 6.4 The steady state solution $y_p(t)/P(\omega)$.

If $\zeta < 0$, the general solution $y(t)$ will increase without bound as time increases, and in physical problems this behavior is called **instability**. In effect, when $\zeta < 0$, energy is fed into the system as time increases, instead of it being removed by friction.

instability, amplification factor, and resonance

The amplitude of the steady state solution is

$$P(\omega) = \frac{A}{[(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2]^{1/2}}, \tag{42}$$

and $P(\omega)/A = [(\Omega^2 - \omega^2) + 4\zeta^2\omega^2]^{-1/2}$ is called the **amplification factor**, because it is the ratio of the amplitude of the solution (*response*) to the amplitude of the forcing function (*input*). The amplification factor attains its maximum value $P_{\max}$ when $\omega = \omega_c$, with $\omega_c^2 = \Omega^2 - 2\zeta^2$, in which case

$$P_{\max} = \frac{A}{2\varsigma(\Omega^2 - \zeta^2)^{1/2}}. \tag{43}$$

The angular frequency ω_c is called the **resonant angular frequency** of the system that is said to experience **resonance** at the the frequency ω_c. It is to avoid exciting resonance that troops marching across a flexible suspension bridge are told to break step. Conversely, it is for this same reason that when one pushes a swing, successive pushes need to be synchronized with each oscillation if the amplitude of the motion is to be built up. If $\zeta = 0$, result (42) shows that resonance occurs when $\omega = \Omega$, leading to an infinite amplification factor. The critical role played by damping in limiting the amplitude of oscillations can be seen from (43).

Figures 6.5a and 6.5b show the variation of the scaled amplification factor $\Omega^2 P(\omega)/A$ and the phase angle ϕ as functions of ω/Ω for a range of values of ζ/Ω.

Care must always be exercised when finding the phase angle ϕ, because the phase is required to lie in the interval $0 < \phi < \pi$, though the usual domain of definition of the inverse tangent function is $(-\pi/2, \pi/2)$.

The most extreme effect of resonance occurs when there is no damping ($\zeta = 0$), though this can never happen in physical problems because there are always some dissipative effects. In the absence of damping, the **natural angular frequency** of oscillations is Ω, and equation (42) shows that when the vibrations are forced by a

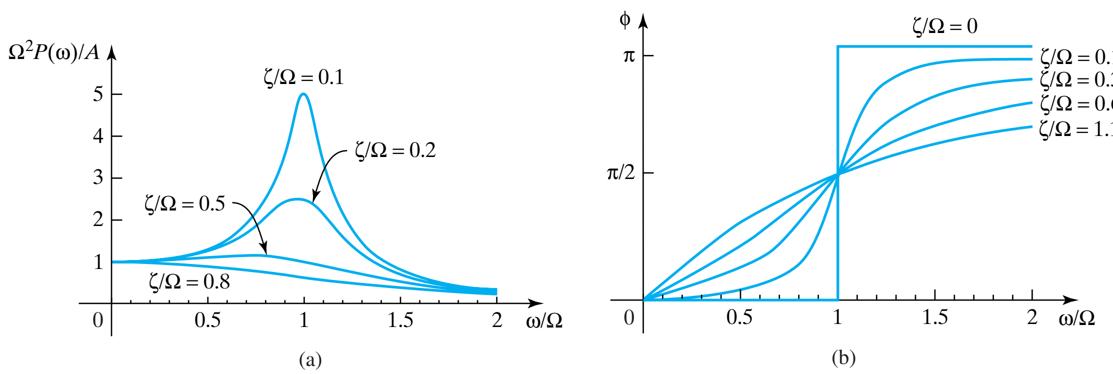


FIGURE 6.5 (a) Amplitude as a function of ω/Ω. (b) Phase angle as a function of ω/Ω.

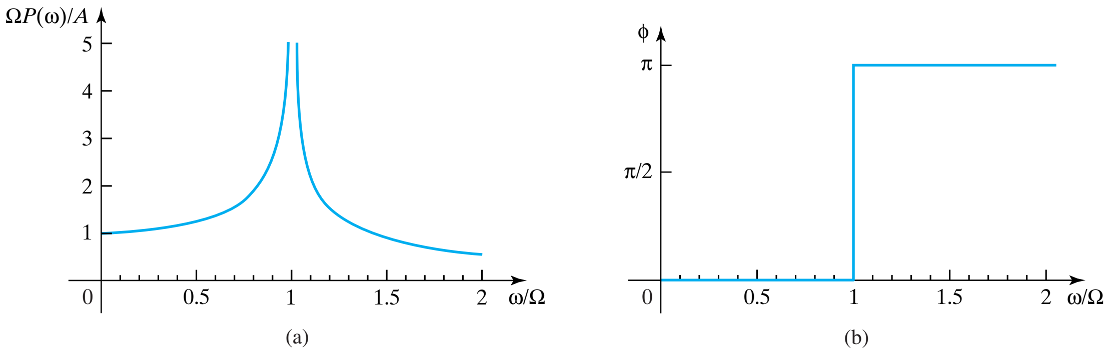


FIGURE 6.6 (a) Variation of amplitude. (b) Variation of phase.

sinusoidal input of angular frequency ω, the amplitude of the steady state solution is

$$P(\omega) = \frac{A}{|\Omega^2 - \omega^2|^{1/2}}.$$

This shows that $P(\omega)$ becomes infinite when the exciting angular frequency ω equals the natural angular frequency Ω. The variation of $\Omega P(\omega)/A$ as ω/Ω varies is shown in Fig. 6.6a, while the corresponding variation of the phase is shown in Fig. 6.6b for the limiting case $\omega \to \Omega$.

To understand how the solution becomes unstable when $\omega = \Omega$, it is necessary to consider the solution of

$$\frac{d^2y}{dt^2} + \Omega^2 y = A \sin \Omega t, \quad \text{with } y(0) = 0, (dy/dt) = 0, t = 0.$$

We find that

$$y(t) = \frac{A}{2\Omega^2}(\sin \Omega t - \Omega t \cos \Omega t),$$

and the variation of $y(t)$ is shown in Fig. 6.7, from which it can be seen that when the damping is zero, forcing at the resonant angular frequency causes the amplitude of the oscillations to grow linearly with time.

beats

An interesting and important property of oscillatory solutions under conditions that allow dissipation to be ignored is to be found in the occurrence of **beats** in the

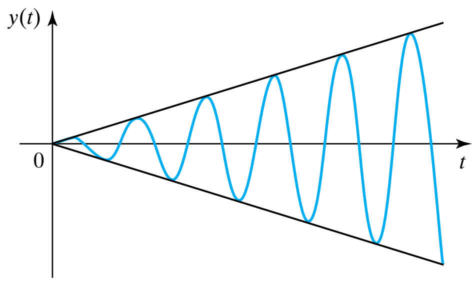


FIGURE 6.7 Linear growth of amplitude with time when $\zeta = 0$.

steady state solution. Consider a solution of the form

$$y(t) = \frac{A}{|\Omega^2 - \omega^2|^{1/2}}[\cos \omega t - \cos \Omega t].$$

Subtracting the trigonometric identities $\cos(C - D) = \cos C \cos D + \sin C \sin D$ and $\cos(C + D) = \cos C \cos D - \sin C \sin D$, and then setting $C = (\Omega + \omega)/2$ and $D = (\Omega - \omega)/2$, gives

$$\cos \omega t - \cos \Omega t = 2 \sin\left(\frac{(\Omega + \omega)t}{2}\right) \sin\left(\frac{(\Omega - \omega)t}{2}\right),$$

so the solution becomes

$$y(t) = \frac{2A}{|\Omega^2 - \omega^2|^{1/2}} \sin\left(\frac{(\Omega + \omega)t}{2}\right) \sin\left(\frac{(\Omega - \omega)t}{2}\right).$$

This result can be written

$$y(t) = E(t) \sin\left(\frac{(\Omega + \omega)t}{2}\right), \quad \text{with } E(t) = \frac{2A}{|\Omega^2 - \omega^2|^{1/2}} \sin\left(\frac{(\Omega - \omega)t}{2}\right),$$

showing that when ω is close to Ω, the solution is in the form of a component with the "high angular frequency" $(\Omega + \omega)/2$, modulated by an amplitude

$$E(t) = \frac{2A}{|\Omega^2 - \omega^2|^{1/2}} \sin\left(\frac{(\Omega - \omega)t}{2}\right);$$

with the "low angular frequency" $(\Omega - \omega)/2$.

This solution is seen to be in the form of "pulses" at the higher angular frequency $(\Omega + \omega)/2$ modulated by the lower angular frequency $(\Omega - \omega)/2$. A typical physical example of *beats* can be experienced when listening to two sound waves with similar frequencies Ω_1 and Ω_2 that interact. Then, provided the amplitudes are similar, the sound at the higher frequency is heard as pulses that arrive at the lower frequency. Figure 6.8 shows a typical situation where beats occur, and when listening to such interacting sound waves the high frequency would be heard as a slow throbbing sound.

EXAMPLE 6.6 Solve the initial value problem

$$4\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + 37y = 12 \cos t, \quad \text{with } y(0) = 1,\ y'(0) = -2.$$

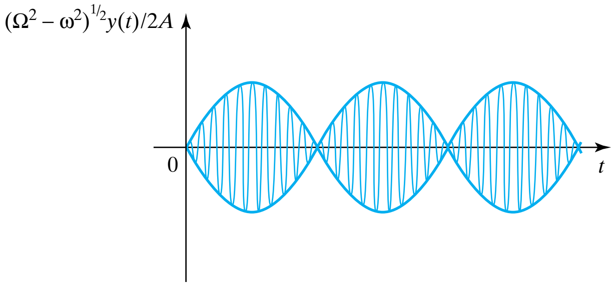


FIGURE 6.8 The phenomenon of beats produced when frequencies ω and Ω are close.

an example showing the makeup of a typical solution

Solution The characteristic equation is

$$4\lambda^2 + 4\lambda + 37 = 0,$$

with the roots $\lambda_1 = -(1/2) + 3i$ and $\lambda_2 = -(1/2) - 3i$, so the complementary function is

$$y_c(t) = \exp[-t/2](C_1 \sin 3t + C_2 \cos 3t).$$

When written in the standard form the differential equation becomes

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{37}{4}y = 3\cos t.$$

Comparison with (33) shows that $\zeta = 1/2$, $\Omega^2 = 37/4$, $A = 3$, and $\omega = 1$, so $\omega_0 = (\Omega^2 - \zeta^2)^{1/2} = 3$. Substituting these results into equations (35) gives $C = 48/1105$ and $D = 396/1105$, so the general solution is

$$y(t) = \exp[-t/2](C_1 \sin 3t + C_2 \cos 3t) + \frac{48}{1105}\sin t + \frac{396}{1105}\cos t.$$

Imposing the initial condition $y(0) = 1$ on $y(t)$ gives

$$1 = C_2 + (396/1105), \quad \text{so} \quad C_2 = 709/1105.$$

Similarly, imposing the initial condition $y'(0) = -2$ on $y(t)$ gives

$$-2 = 48/1105 - (1/2)C_2 + 3C_1, \quad \text{so} \quad C_1 = -1269/2210.$$

Finally, substituting the values of C_1 and C_2 into the general solution shows that the solution of the initial value problem is

$$y(t) = \frac{1}{2210}\exp(-t/2)(1418\cos 3t - 1269\sin 3t) + \frac{1}{1105}(48\sin t + 396\cos t).$$

The steady state solution is

$$y_p(t) = \frac{1}{1105}(48\sin t + 396\cos t) = \frac{12}{\sqrt{1105}}\cos(t - \phi),$$

where the phase lag $\phi = \arctan C/D = \arctan(48/396) = 0.1206$ radians, and the transient solution is

$$y_c(t) = \frac{1}{2210}\exp(-t/2)(1418\cos 3t - 1269\sin 3t).$$

On the following page, the transient solution $y_p(t)$ is shown in Fig. 6.9a, the steady state solution $y_c(t)$ in Fig. 6.9b, and the complete solution $y(t)$ of the initial value problem in Fig. 6.9c. ■

Summary

This section showed that the solution of a nonhomogeneous constant coefficient equation where the independent variable is the time comprises two parts: one called the transient solution, which describes the startup of the solution, and another called the steady state solution, which describes the nature of the time-dependent solution that remains when the transient solution has decayed sufficiently to become negligible.

The important case involving a sinusoidal forcing function was examined in detail, and the terms amplitude, frequency, and phase angle of the solution were explained, together with the important effect of resonance.

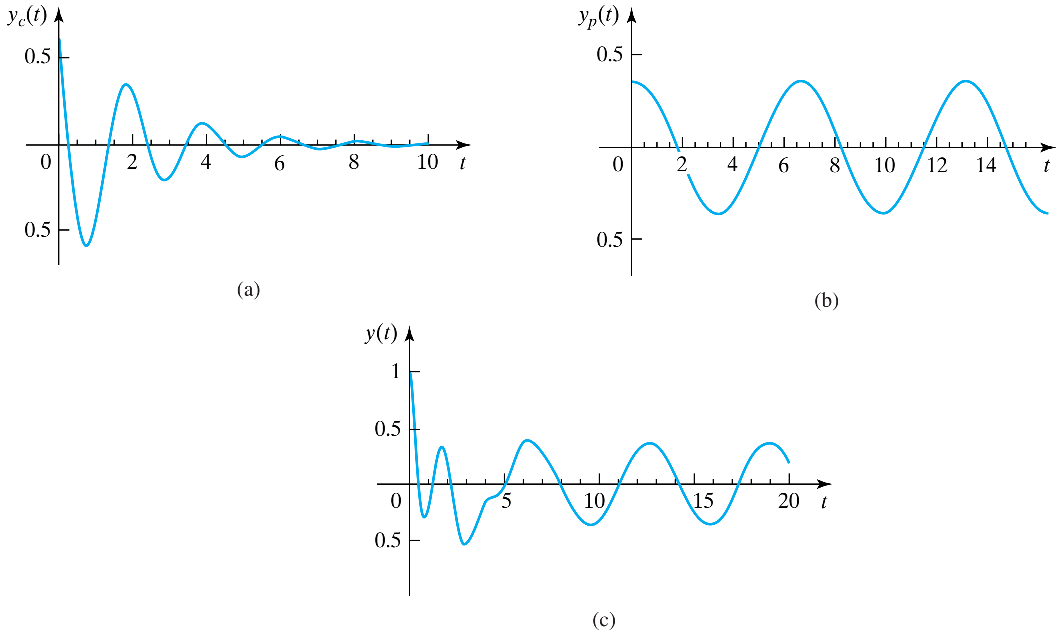


FIGURE 6.9 (a) The transient solution. (b) The steady state solution. (c) The complete solution.

EXERCISES 6.2

In Exercises 1 through 7 solve the initial value problem using the methods of this section, and identify the steady state and transient solutions. Confirm the results for the even numbered problems by computer algebra and plot their solutions for some interval $0 < t < T$.

1. $y'' + 2y' + 5y = 2\sin x$, with $y(0) = 1$, $y'(0) = 0$.
2. $y'' + 2y' + 5y = 3\sin x$, with $y(0) = 0$, $y'(0) = 0$.
3. $y'' + 2y' + y = \sin x$, with $y(0) = 1$, $y'(0) = 0$.
4. $y'' + 2y' + y = \sin 2x$, with $y(0) = 2$, $y'(0) = 0$.
5. $y'' + 3y' + 2y = \sin 3x$, with $y(0) = 0$, $y'(0) = 1$.
6. $y'' + 2y' + 5y = \sin x$, with $y(0) = 0$, $y'(0) = 1$.
7. $y'' + 5y' + 6y = A\sin x$, with $y(0) = 3$, $y'(0) = 1$.
8. Use the argument in Section 6.2 when establishing the results in (35) to show that if the forcing function on the right of (33) is replaced by $A\sin\omega t$, and the particular integral is written

$$y_p(t) = C\sin\omega t + D\cos\omega t,$$

the constants C and D are given by

$$C = \frac{A(\Omega^2 - \omega^2)}{(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2} \quad \text{and} \quad D = \frac{2\zeta\omega A}{(\Omega^2 - \omega^2)^2 + 4\zeta^2\omega^2},$$

and that the phase angle ϕ is such that

$$\tan\phi = -\frac{2\zeta\omega}{\Omega^2 - \omega^2}.$$

In Exercises 9 through 14 use the results of Exercise 8 when solving the initial value problem. Find the phase angle and identify the steady state and transient solutions.

9. $y'' + 5y' + 6y = 2\cos x$, with $y(0) = 1$, $y'(0) = 1$.
10. $y'' + 7y' + 6y = 2\cos 3x$, with $y(0) = 2$, $y'(0) = 1$.
11. $y'' + 6y' + 9y = 2\cos 3x$, with $y(0) = 2$, $y'(0) = 2$.
12. $y'' + 2y' + 2y = \cos 4x$, with $y(0) = 0$, $y'(0) = 2$.
13. $y'' + 6y' + 8y = 3\cos 2x$, with $y(0) = 4$, $y'(0) = 1$.
14. $y'' + 2y' + 5y = 3\cos 3x$, with $y(0) = 2$, $y'(0) = 3$.
15. The fall of a loaded parachute is determined by the differential equation

$$m\frac{d^2y}{dt^2} + kg\frac{dy}{dt} + mg = 0,$$

where m is the weight of the payload in pounds, k is the drag coefficient of the parachute, $y(t)$ is its height above the ground at time t in seconds, and g is the acceleration due to gravity. Taking $g = 32$ ft/sec^2, $k = 10$ lb/ft/sec, and the initial speed of fall at time $t = 0$ when the parachute opens 2000 ft above the ground

to be $dy/dt = -32$ ft/sec (remember $y(t)$ is measured upward but the speed is downward), find $y(t)$ and the speed of fall at time t as functions of m. Use the result to find the largest payload M in pounds if the speed of fall on landing is not to exceed 24 ft/sec. Plot $y(t)$ for $m = M$ and estimate the time of descent in this case.

16. Stokes' law $F = 6\pi a\eta u$ determines the drag F on a sphere of radius a moving slowly through a fluid with viscosity η at a speed u. Let the density of the sphere be ρ_1 and the fluid density be ρ_2 $(\rho_1 > \rho_2)$. Find the equation of motion of the sphere in terms of the distance $x(t)$ from its point of release, if it falls from rest in the fluid at time $t = 0$. Solve the equation of motion to find $x(t)$, and hence the speed of fall, as functions of time. Suggest how this result could be used to determine the viscosity of oil in an experiment involving the release from rest of a ball bearing that is allowed to fall vertically through oil contained in a long glass cylinder.

17. A spherical container of radius a and density ρ_1 is released from rest on the sea bed at a depth h below the surface and allowed to float slowly upward in still water of density ρ_2, where $\rho_2 - \rho_1$ is small and $\rho_2 > \rho_1$. Assuming that Stokes' law in Exercise 16 applies, and the viscosity of the water is η, find the distance $x(t)$ of the container from the sea bed as a function of time, and use it to write down the equation determining the time T when the container reaches the surface. Estimate this time, and suggest how a more accurate value of T could be obtained.

18. As $\omega \to 1$, from either above or below, so the solution $x(t)$ of $x'' + x = \sin \omega t$ subject to the initial conditions $x(0) = x'(0) = 0$ tends to the divergent resonance solution illustrated in Fig. 6.7. Use computer algebra to plot the solution for $\omega = 0.85$, 0.95, 0.99, 1.0, 1.05, and 1.1 to illustrate how the amplitude of the oscillations tends to a linear growth as $\omega \to 1$. Show that for $\omega = 1$, $x = \frac{1}{2}(\sin t - t\cos t)$.

19. Typically, beats occur when two slowly varying oscillations with equal amplitudes and almost equal frequencies are superimposed. Use computer algebra to plot $x(t) = \cos \omega_1 t + \cos \omega_2 t$, with suitable values of ω_1 and ω_2 and a sufficiently long time interval $0 \le t \le T$, to show a clear pattern of the beats. Find the equation determining the high-frequency oscillation and the equations forming the envelope of the high-frequency component.

6.3 Homogeneous Linear Higher Order Constant Coefficient Equations

A Typical Example Leading to a Fourth Order System

Linear nth order constant coefficient differential equations often arise as a result of the elimination of all but one of the unknowns in a system of simultaneous lower order differential equations. To see how this can happen, consider the longitudinal motion of three equal particles of mass m, coupled together by four identical springs each of unstrained length l with spring constant k, with the left and right ends of the system clamped, as illustrated in Fig. 6.10.

Now suppose that the system oscillates in the direction of the springs, with y_1, y_2, and y_3 the displacements of the masses from their equilibrium positions, as shown in Fig. 6.10. Equating the rate of change of momentum $d/dt\{m(dy_1/dt)\}$ of the mass with coordinate y_1 to the sum of the restoring force $k(y_2 - y_1)$ due to the second spring and the force $k(y_3 - y_1)$ due to the third spring shows that the equation of motion of the first mass is

$$m\frac{d^2y_1}{dt^2} = k(l - y_1) + k(y_2 - y_1 - l) = k(y_2 - 2y_1).$$

Similar arguments applied to the second and third masses in this system with *three degrees of freedom* (the coordinates y_1, y_2, and y_3) gives the other two coupled equations of motion,

$$m\frac{d^2y_2}{dt^2} = k(l + y_1 - y_2) + k(y_3 - y_2 - l) = k(y_1 + y_3 - 2y_2)$$

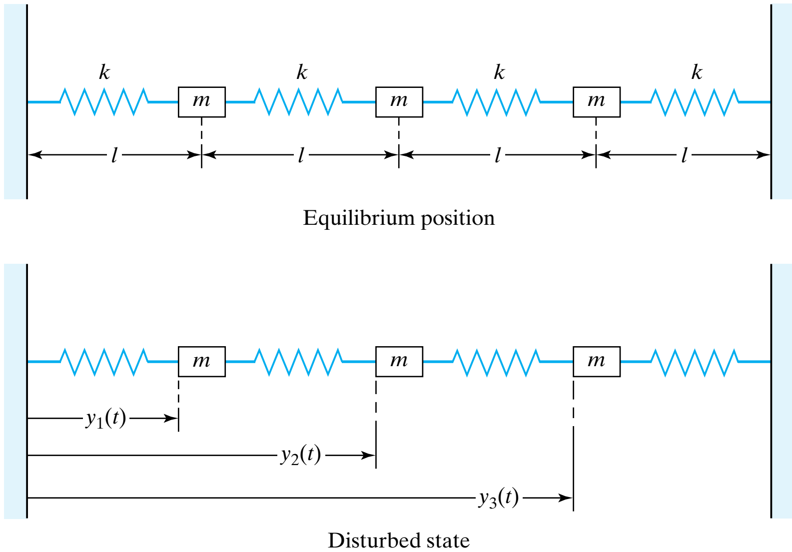


FIGURE 6.10 A three-mass–spring system with its ends clamped.

and

$$m\frac{d^2y_3}{dt^2} = k(l + y_2 - y_3) + k(4l - y_3 - l) = k(4l + y_2 - 2y_3).$$

Eliminating any two of the three unknowns y_1, y_2, and y_3 from these three equations of motion leads to a homogeneous sixth order constant coefficient differential equation for the remaining unknown. Initial conditions for the system are the values $y_i(0)$ and $y_i'(0)$ for $i = 1, 2,$ and 3.

More complicated systems of this type are used to study one-dimensional waves in various types of periodic structure ranging from chains of low-pass electrical filters to the vibration of molecules in crystal lattices.

A different example that gives rise to a fourth order differential equation is the modeling of a two degree of freedom vibration damper for a motor generator of mass M. Unless damped, the vertical vibrations due to the periodic motion of the pistons are passed to the foundations of the building and can cause unacceptable vibrations throughout the building. One way of dealing with this problem is not only to mount the motor generator on a spring and damper system, but also to spring mount a smaller mass m on top of the motor generator, as in Fig. 6.11, and to adjust the two spring constants and the mass m so that the vertical oscillations of M are minimized and passed instead to the smaller mass m mounted on the motor generator.

Let the mass M be connected to the foundation by a spring with spring constant K, and let the spring constant of the spring supporting mass m be k. To make the model more realistic, suppose that in addition there is a viscous damper fitted between the mass M and the foundation that exerts a resistance proportional to the speed of its displacement with constant of proportionality μ, and let the displacements of the masses M and m from their equilibrium positions be x and y, respectively. Suppose also that the vibrational force acting on M due to the operation of the motor generator is $F(t)$.

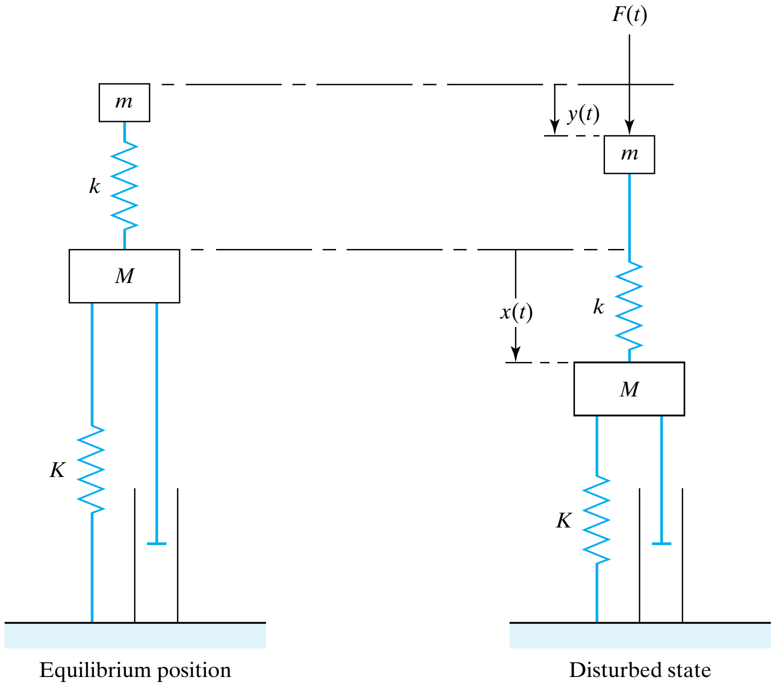


FIGURE 6.11 A two degree of freedom vibration system with a viscous damper.

The equation of motion of the mass M obtained by equating its rate of change of momentum to the combined restoring forces of the two springs, the viscous damper, and the vibrational force $F(t)$ is

$$M\frac{d^2x}{dt^2} = -k(x-y) - Kx - \mu\frac{dx}{dt} + F(t),$$

and the equation of motion of the mass m obtained by equating its rate of change of momentum to the restoring force exerted by the top spring is

$$m\frac{d^2y}{dt^2} = -k(y-x).$$

Eliminating y between these two equations gives the fourth order constant coefficient equation for x

$$\frac{d^4x}{dt^4} + \alpha\frac{d^3x}{dt^3} + (\beta+\gamma+\gamma\delta/\beta)\frac{d^2x}{dt^2} + \alpha\beta\frac{dx}{dt} + \gamma\delta x = \frac{1}{M}\left(\gamma F(t) + \frac{d^2F}{dt^2}\right),$$

where $\alpha = \mu/M$, $\beta = k/m$, $\gamma = k/M$, and $\delta = K/m$.
Similarly, eliminating x between the two equations gives

$$\frac{d^4y}{dt^4} + \alpha\frac{d^3y}{dt^3} + (\beta+\gamma+\gamma\delta/\beta)\frac{d^2y}{dt^2} + \alpha\beta\frac{dy}{dt} + \gamma\delta y = \frac{\gamma}{M}F(t).$$

When $F(t)$ is a periodic force with frequency ω, and the constants k, K, and m are adjusted to take account of resonance in the spring and damper mounting, the system can be tuned so that the displacement $x(t)$ is reduced almost to zero, and the vibration is transferred instead to the mass m mounted on top of the motor generator.

General Homogeneous Higher Order Constant Coefficient Equations

The homogeneous **linear constant coefficient *n*th order equation**

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \tag{44}$$

has properties that are similar to those of second order equations.

If $y_1(x), y_2(x), \ldots, y_r(x)$ are any r solutions of (44), the linearity of the equation means that the linear combination of functions

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_r y_r(x),$$

linear superposition in higher order systems

with $c_1, c_2, \ldots, c_r$ arbitrary constants, is also a solution. This **linear superposition** property of solutions of the homogeneous equation is an extension of the same property encountered in Section 6.1 when considering homogeneous constant coefficient second order equations. The proof of this property follows by substituting $y(x)$ into the left-hand side of (44), using the linearity of the differentiation operation

$$\frac{d^s}{dx^s}(c_1 y_1 + c_2 y_2 + \cdots + c_r y_r) = c_1 \frac{d^s y_1}{dx^s} + c_2 \frac{d^s y_2}{dx^s} + \cdots + c_r \frac{d^s y_r}{dx^s},$$

for $s = 0, 1, \ldots, n$, where $d^0 y/dx^0 \equiv y$ and grouping terms, to obtain r expressions of the form

$$c_i\left(\frac{d^n y_i}{dx^n} + a_1 \frac{d^{n-1} y_i}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy_i}{dx} + a_n y_i\right).$$

Each of these expressions vanishes, because the $y_i(x)$ are solutions of the homogeneous equation, so the result of substituting $y(x)$ into the left side of (44) is to reduce it to zero, showing that

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_r y_r(x)$$

is a solution.

basis, solution space, and general solutions

It will be shown later that the homogeneous equation (44) has n linearly independent solutions $y_1(x), \ldots, y_n(x)$, and that these form a **basis** for its **solution space**. This means that every particular solution of (44) can be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x), \tag{45}$$

for some choice of constants $c_1, c_2, \ldots, c_n$. It is because of this property that (45) is called the **general solution** of (44).

A more general test for linear independence than the one in Section 6.1 is needed to ensure that the n solutions $y_1(x), y_2(x), \ldots, y_n(x)$ of (44) form a basis for the solution space. To obtain this test we must first extend the earlier definition of linear independence in a natural way to a set of functions $g_1(x), g_2(x), \ldots, g_n(x)$ defined over an interval $a \le x \le b$. The set of functions will be said to be **linearly**

independent over the interval if for all x in the interval,

linear independence and dependence

$$k_1g_1(x) + k_2g_2(x) + \cdots + k_ng_n(x) = 0 \tag{46}$$

is only true if $k_1 = k_2 = \cdots = k_n = 0$; otherwise, the set of functions will be said to be **linearly dependent**.

As the test will be needed later for solutions of linear differential equations more general than (44), it will be derived for the variable coefficient differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0, \tag{47}$$

where the coefficients $a_i(x)$ are continuous functions of x for $a \le x \le b$. The test will also apply to solutions of (44), because a constant is a special case of a continuous function.

The derivation starts from the fact that if the functions $y_1(x), y_2(x), \ldots, y_n(x)$ are solutions of the nth order equation (47) with continuous coefficients over an interval $a \le x \le b$, then they will be everywhere continuous and differentiable at least $n-1$ times over this same interval. By definition, the functions will be linearly independent over the interval $a \le x \le b$ if the equation

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_n(x)y_n(x) = 0 \tag{48}$$

is only true if $c_1 = c_2 = \cdots = c_n = 0$ for all x in the interval. Differentiating the equation $n-1$ times gives

$$\begin{aligned} c_1y_1(x) + c_2y_2(x) + \ldots + c_n(x)y_n(x) &= 0 \\ c_1y_1^{(1)}(x) + c_2y_2^{(1)}(x) + \cdots + c_ny_n^{(1)}(x) &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ c_1y_1^{(n-1)}(x) + c_2y_2^{(n-1)}(x) + \cdots + c_ny_n^{(n-1)}(x) &= 0. \end{aligned} \tag{49}$$

This homogeneous system of equations can only have the null solution $c_1 = c_2 = \cdots = c_n = 0$ that is necessary to ensure the linear independence of the functions $y_1(x), y_2(x), \ldots, y_n(x)$ if the determinant W of the coefficients is nonvanishing, for $a \le x \le b$. This shows that the required condition for linear independence is $W \ne 0$, for $a \le x \le b$, where

Wronskian determinant

$$W = \begin{vmatrix} y_1(x) & y_2(x) & \ldots & y_n(x) \\ y_1^{(1)}(x) & y_2^{(1)}(x) & \ldots & y_n^{(1)}(x) \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots & y_n^{(n-1)}(x) \end{vmatrix}. \tag{50}$$

The determinant W is called the **Wronskian** of the set of functions $y_1(x)$, $y_2(x), \ldots, y_n(x)$, and it is named after the Polish mathematician who introduced the condition. We have proved the following theorem concerning the linear independence of solutions of homogeneous linear differential equations with continuous coefficients.

Jozef Maria Wronski (1778–1853)
A Polish philosopher and mathematician now remembered only because of his introduction of the functional determinant called the Wronskian.

THEOREM 6.2

the Wronskian test for linear independence

Wronskian test for linear independence Let $y_1(x), y_2(x), \ldots, y_n(x)$ be $n-1$ times differentiable solutions of a homogeneous linear nth order differential equation with continuous coefficients that is defined over an interval $a \le x \le b$. Then a necessary and sufficient condition for the functions to be linearly independent solutions of the differential equation is that their Wronskian W is nonvanishing over this interval. The solutions will be linearly dependent over the interval if W vanishes identically. ■

EXAMPLE 6.7

(a) The set of continuous functions $\cosh x$, $\sinh x$, 1 is linearly independent, because the Wronskian

$$W = \begin{vmatrix} \cosh x & \sinh x & 1 \\ \sinh x & \cosh x & 0 \\ \cosh x & \sinh x & 0 \end{vmatrix} = \sinh^2 x - \cosh^2 x = -1, \quad \text{for all } x.$$

(b) The set of continuous functions $1, x, x^2, (1+x)^2$ is linearly dependent because the Wronskian

$$W = \begin{vmatrix} 1 & x & x^2 & (1+x)^2 \\ 0 & x & 2x & 2+2x \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \quad \text{for all } x.$$

This result is obvious without appeal to Theorem 6.2, because setting $y_1 = 1$, $y_2 = x$, $y_3 = x^2$, and $y_4 = (1+x)^2$, we have $y_4 = y_1 + 2y_2 + y_3$, showing that y_4 is a linear combination of y_1, y_2, and y_3. ■

It should be understood that when Theorem 6.2 is used as a general test for the linear independence of an arbitrary set of functions $u_1, u_2, \ldots u_n$ defined over an interval I, the vanishing of their Wronskian is a *necessary* condition for their linear independence over the interval, but it is *not* a sufficient condition if any of the functions involved are discontinuous within the interval.

It is the requirement in Theorem 6.2 that the functions be solutions of a homogeneous linear differential equation with *continuous* coefficients that ensures that the vanishing of the Wronskian is both a necessary and sufficient condition for their linear independence, though the details of the proof of this are omitted.

initial value problem and initial conditions

An **initial value problem** for the nth order linear differential equations (44) and (47) at a point $x = x_0$ involves specifying the **initial conditions** $y(x_0) = k_0$, $y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_n$ for $y(x)$, and its first $n-1$ derivatives at the point x_0, where the constants $k_1, k_2, \ldots, k_n$ can be specified arbitrarily. The derivative $y^{(n)}(x_0)$ cannot be specified as an initial condition, because it is determined by the differential equation itself once the stated initial conditions have been given.

The following is the fundamental existence and uniqueness theorem for linear higher order differential equations.

THEOREM 6.3

Existence and uniqueness of solutions Let the coefficients of the homogeneous differential equation (47) be continuous functions over an interval $a < x < b$ that

contains the point x_0 and $a_0(x) \neq 0$ in (a, b). Then a unique solution exists on this interval that satisfies the initial conditions

existence and uniqueness of solutions

$$y(x_0) = k_0, \quad y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_n.$$

Proof A proof of the existence of solutions of initial value problems for linear higher order variable coefficient differential equations is beyond the level of this first account, and so will be omitted. However, the existence and uniqueness of solutions of initial value problems for constant coefficient equations will follow from the subsequent work in which the form of the general solution will be found and its constants matched so that it satisfies the initial conditions.

It remains for us to establish the uniqueness of the initial value problem for linear higher order variable coefficient equations with continuous coefficients. Let us consider equation (47), and write its general solution

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

Differentiating this result $n - 1$ times, and after each differentiation substituting the initial conditions, leads to the following system of simultaneous equations:

$$\begin{aligned}
&c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_n(x) y_n(x_0) = k_0 \\
&c_1 y_1^{(1)}(x_0) + c_2 y_2^{(1)}(x_0) + \cdots + c_n y_n^{(1)}(x_0) = k_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = k_{n-1}.
\end{aligned}$$

This nonhomogeneous system of linear equations will have a unique solution for the constant coefficients $c_1, c_2, \ldots, c_n$ provided the determinant of the coefficients does not vanish. The determinant is simply the Wronskian $W(x_0)$, and by hypothesis the n solutions are linearly independent, so $W(x_0) \neq 0$ for $a \leq x \leq b$. Consequently, the coefficients $c_1, c_2, \ldots, c_n$ are uniquely determined and, when substituted into the general solution, lead to a unique solution of the initial value problem. ■

To solve the homogeneous constant coefficient equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0, \tag{51}$$

we proceed as with a second order equation and seek solutions of the form $y(x) = ce^{\lambda x}$, with c and λ constants. Substituting $y(x)$ into (51) leads to the result

$$(\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n) e^{\lambda x} = 0,$$

after which cancellation of the nonvanishing factor $e^{\lambda x}$ shows λ may be any of the roots of the **characteristic equation**

characteristic equation for higher order equations

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0. \tag{52}$$

This polynomial of degree n has n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ that either will all be real or, if some are complex, will occur in complex conjugate pairs. To each root λ_i there will correspond a solution $y_i(x)$, and the linearly independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ form a **basis** for the **solution space**. An arbitrary linear combination of the n basis functions forms the **complementary function** for (51).

Rules for constructing the complementary function of an *n*th order constant coefficient differential equation

how to construct the complementary function

The differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

with real coefficients $a_1, a_2, \ldots, a_n$ has the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0,$$

with the n roots $\lambda_1, \lambda_2, \ldots, \lambda_n$.

1. To a single real root $\lambda = \alpha$ there corresponds the single solution $e^{\alpha x}$, with A an arbitrary constant.
2. Substitution shows that to a real root $\lambda = \alpha$ with multiplicity r (repeated r times) there correspond the r linearly independent solutions

$$e^{\alpha x},\ xe^{\alpha x}, \ldots, x^{r-1} e^{\alpha x}.$$

3. To a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$ there correspond the two solutions

$$e^{\alpha x} \cos \beta x \quad \text{and} \quad e^{\alpha x} \sin \beta x.$$

4. To a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$ repeated s times, there correspond the $2s$ solutions

$$\begin{aligned} &e^{\alpha x} \cos \beta x,\ e^{\alpha x} \sin \beta x,\ e^{\alpha x} x \cos \beta x,\ e^{\alpha x} x \sin \beta x, \ldots \\ &\quad \ldots, e^{\alpha x} x^{s-2} \cos \beta x,\ e^{\alpha x} x^{s-2} \sin \beta x,\ e^{\alpha x} x^{s-1} \cos \beta x, \\ &\quad e^{\alpha x} x^{s-1} \sin \beta x. \end{aligned}$$

5. The general solution of the differential equation is an arbitrary linear combination of all solutions produced by the preceding rules.

To see why the functions in Rules 2 and 4 are solutions of the differential equation, we consider a typical case in which the differential equation has a real root $\lambda = \mu$ with multiplicity 2. Removing the factor $(\lambda - \mu)^2$ from the characteristic polynomial allows it to be written

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = (\lambda - \mu)^2 \mathbf{Q}(\lambda),$$

where $\mathbf{Q}(\lambda)$ is a polynomial of degree $n - 2$ in λ that does not vanish when $\lambda = \mu$.

Differentiating this result with respect to λ gives

$$n\lambda^{n-1} + (n-1)a_1 \lambda^{n-2} + \cdots + a_{n-1} = 2(\lambda - \mu)\mathbf{Q}(\lambda) + (\lambda - \mu)^2 \mathbf{Q}'(\lambda),$$

and setting $\lambda = \mu$ reduces this to

$$n\mu^{n-1} + (n-1)a_1 \mu^{n-2} + \cdots + a_{n-1} = 0.$$

As the multiplicity of the root is 2, and $e^{\mu x}$ is known to be a solution, it is necessary to show that $xe^{\mu x}$ is also a solution. This will follow if when $xe^{\mu x}$ is substituted into the differential equation the result becomes an identity.

Setting $y(x) = xe^{\mu x}$ and differentiating m times gives $y^{(m)} = m\mu^{m-1}e^{\mu x} + \mu^m xe^{\mu x}$. Substituting this into the left-hand side of the differential equation leads to the result

$$(n\mu^{n-1} + (n-1)a_1\mu^{n-2} + \cdots + a_{n-1})e^{\mu x} + (\mu^n + a_1\mu^{n-1} + \cdots + a_{n-1}\mu + a_n)xe^{\mu x},$$

but this is zero because we have shown that the coefficient of $e^{\mu x}$ is zero, and the coefficient of $xe^{\mu x}$ vanishes because μ is a root of the characteristic equation. Thus, $xe^{\mu x}$ satisfies the differential equation identically and so is a solution. The functions $e^{\mu x}$ and $xe^{\mu x}$ are linearly independent because they are not proportional.

The same form of argument can be extended to the case when $\lambda = \mu$ is a real root of arbitrary multiplicity, whereas the linear independence of the solutions follows from Theorem 6.2. A similar argument can be used when a pair of complex conjugate roots occurs with arbitrary multiplicity, though the details of these extensions are left as exercises.

EXAMPLE 6.8

some typical examples

Find the general solution of

(i) $y''' - 2y'' - 5y' + 6y = 0$;
(ii) $y''' + 2y'' + 4y' = 0$;
(iii) $y^{(\mathrm{iv})} + y'' - 2y = 0$.

Solution

(i) The characteristic equation is

$$\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0.$$

Inspection shows that $\lambda = 1$ is a root, so dividing the characteristic equation by the factor $(\lambda - 1)$ shows that the other two roots are the solutions of $\lambda^2 - \lambda - 6 = 0$, which are $\lambda = -2$ and $\lambda = 3$. Thus, from Rule 1 the general solution is

$$y(x) = C_1 e^x + C_2 e^{-2x} + C_3 e^{3x}.$$

(ii) The characteristic equation is

$$\lambda^3 + 2\lambda^2 + 4\lambda = 0 \quad \text{or} \quad \lambda(\lambda^2 + 2\lambda + 4) = 0,$$

from which we see that $\lambda = 0$, or $\lambda = -1 \pm i\sqrt{3}$.
Combining Rules 1 and 3 shows the general solution to be

$$y(x) = C_1 + e^{-x}(C_2\cos(x\sqrt{3}) + C_3\sin(x\sqrt{3})).$$

(iii) The characteristic equation is

$$\lambda^4 + \lambda^2 - 2 = 0.$$

This is a biquadratic equation, so if we set $m = \lambda^2$, this becomes $m^2 + m - 2 = 0$, with the solutions $m = -2$ and $m = 1$. Thus, λ can take the values $1, -1, i\sqrt{2}$, and $-i\sqrt{2}$. Combining Rules 1 and 3 shows the general solution to be

$$y(x) = C_1e^x + C_2e^{-x} + C_3\cos(x\sqrt{2}) + C_4\sin(x\sqrt{2}).$$ ■

EXAMPLE 6.9 Find the general solution of a homogeneous equation with the characteristic equation

$$\lambda^3(\lambda+4)^2(\lambda^2+2\lambda+5)^2 = 0.$$

Solution In this equation the real root $\lambda = 0$ occurs with multiplicity 3, the real root $\lambda = -4$ occurs with multiplicity 2, and the pair of complex conjugate roots $(\lambda = -1+2i)$ and $(\lambda = -1-2i)$ occur with multiplicity 2.

The terms to be included in the general solution corresponding to the repeated root $\lambda = 0$ follow by setting $\lambda = 0$ and $r = 3$ in Rule 2 to obtain

$$D_1 + D_{2x} + D_3x^2.$$

Similarly, the terms to be included corresponding to the repeated root $\lambda = -4$ follow by setting $\alpha = -4$ and $r = 2$ in Rule 2 to obtain

$$K_1e^{-4x} + K_2xe^{-4x},$$

where K_1 and K_2 are arbitrary constants.

Finally, the terms to be included because of the repeated complex conjugate roots follow by setting $\alpha = -1$, $\beta = 2$, and $s = 2$ in Rule 4 to obtain

$$e^{-x}\{E_1\cos 2x + F_1\sin 2x + E_2x\cos 2x + F_2x\sin 2x\}.$$

Collecting terms shows that the general solution is

$$y(x) = D_1 + D_{2x} + D_3x^2 + K_1e^{-4x} + K_2xe^{-4x}$$
$$+e^{-x}\{E_1\cos(2x) + F_1\sin(2x) + E_2x\cos(2x) + F_2x\sin(2x)\}.$$

This general solution contains nine arbitrary constants, as would be expected because the characteristic polynomial is of degree 9. ■

EXAMPLE 6.10 Solve the initial value problem

$$y''' - 2y'' - 5y' + 6y = 0, \quad \text{with } y(0) = 1,\ y'(0) = y''(0) = 0.$$

Solution The general solution was shown in Example 6.8 (i) to be

$$y(x) = C_1e^x + C_2e^{-2x} + C_3e^{3x}.$$

The initial conditions require that

$$\begin{array}{ll} (y(0)=1) & 1 = C_1 + C_2 + C_3 \\ (y'(0)=0) & 0 = C_1 - 2C_2 + 3C_3 \\ (y''(0)=0) & 0 = C_1 + 4C_2 + 9C_3. \end{array}$$

The solution of this system of equations is $C_1 = 1$, $C_2 = 1/5$, $C_3 = -1/5$, so the solution of the initial value problem is

$$y(x) = e^x + \frac{1}{5}e^{-2x} - \frac{1}{5}e^{3x}.$$ ■

EXAMPLE 6.11 Solve the initial value problem

$$y''' + 2y'' + 4y' = 0, \quad \text{with } y(0) = 0,\ y'(0) = 1,\ y''(0) = 0.$$

Solution The general solution was found in Example 6.8 (ii) to be

$$y(x) = C_1 + e^{-x}(C_2\cos(x\sqrt{3}) + C_3\sin(x\sqrt{3})).$$

The initial conditions require that

$$\begin{aligned}
&(y(0) = 0) \quad C_1 + C_2 = 0 \\
&(y'(0) = 1) \quad 1 = -C_2 + C_3\sqrt{3} \\
&(y''(0) = 0) \quad 0 = C_2 + C_3\sqrt{3}.
\end{aligned}$$

These equations have the solution $C_1 = 1/2$, $C_2 = -1/2$, and $C_3 = \sqrt{3}/6$, so the solution is

$$y(x) = \frac{1}{2} + \frac{\sqrt{3}}{6}e^{-x}\sin x\sqrt{3} - \frac{1}{2}e^{-x}\cos x\sqrt{3}. \quad \blacksquare$$

Summary

This section extended the discussion of linear second order constant coefficient equations to higher order equations, and showed how the characteristic equation again determines the nature of the solutions that enter into the complementary function. The concept of linearly independent functions was extended, and it was shown that the set of linearly independent functions associated with a higher order equation forms a basis for its solution space. The Wronskian was defined and shown to provide a test for the linear independence of a set of solutions of a higher order equation. Rules were given for construction of the complementary function of an *n*th order constant coefficient equation, and then applied to some typical examples.

EXERCISES 6.3

1. Use the Wronskian test to prove the linear independence of the functions
e^x, xe^x, x^2e^x for $|x| < \infty$.

2. Use the Wronskian test to prove the linear independence of the functions
$\sin x, e^x\sin x, e^x\cos x$.

3. Test the following functions for linear independence: $3, -x, x^2, (1 + 2x)^2$.

4. Test the following functions for linear independence: $1, \ln x, \ln x^{1/2}, e^x$ for $x = 0$.

In Exercises 5 through 12 show that the given functions form a basis for the associated differential equation. Write down the general solution, state the interval in which it is defined, and, where required, solve the given initial value problem.

5. $xy'' - y' - 4x^3y = 0$; $\cosh x^2$ and $\sinh x^2$.

6. $xy'' - y' + 4x^3y = 0$; $\sin x^2$ and $\cos x^2$.

7. $y''' + 3y'' + 9y' - 13y = 0$; e^x, $e^{-2x}\cos 3x$, $e^{-2x}\sin 3x$.

Solve the initial value problem for which $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 0$.

8. $x^3y''' - x^2y'' + 2xy' - 2y = 0$; $x, x^2, x\ln|x|$.

Solve the initial value problem for which $y(1) = 1$, $y'(1) = 1$, and $y''(1) = 0$.

9. $(8x^2 + 1)y'' - 16xy' + 16y = 0$; $2x, 8x^2 - 1$.

10. $y'' - 16xy' + (64x^2 - 8)y = 0$; $\exp(4x^2), 2x\exp(4x^2)$.

11. $[4 - 2x\cot(x/2)]y'' - xy' + y = 0$; $x/2, \sin(x/2)$.

12. $3x^3y'' + xy' - y = 0$; $3x, 3x\exp[1/(3x)]$.

In Exercises 13 through 18 solve the initial value problems using the five stated rules for the construction of the complementary function and, when available, use computer algebra to check the results.

13. $y''' + y'' - 4y = 0$, with $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$.

14. $y''' + 3y'' - 4y = 0$, with $y(1) = -1$, $y'(1) = 0$, $y''(1) = 1$.

15. $y''' + 3y'' + 7y' + 5y = 0$, with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$.

16. $y''' - 2y'' + 5y' + 26y = 0$, with $y(0) = 0$, $y'(0) = 1$, $y''(0) = 1$.

17. $y^{(iv)} - y'' - 2y = 0$, with $y(0) = 1$, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$.

18. $y^{(iv)} - y'' - 6y = 0$, with $y(0) = 0$, $y'(0) = 1$, $y''(0) = 0$, $y'''(0) = 0$.

19.* A gyrostatic pendulum is a pendulum bob (mass) suspended by a light inextensible string from a fixed point, with the bob allowed to swing around its equilibrium position. If the displacement of the bob from its equilibrium position is small, the x and y coordinates of the bob as a function of time t can be shown to satisfy the

coupled differential equations

$$\frac{d^2x}{dt^2}+a\frac{dy}{dt}+c^2x=0 \quad \text{and} \quad \frac{d^2y}{dt^2}-a\frac{dx}{dt}+c^2y=0,$$

with $a>0$. Find the general solution for $x(t)$ and $y(t)$. By examination of the constants in the general solution identify two situation in which the motion of the bob will be in a circle (a circular pendulum), in each case commenting on the angular velocity of the bob.

20.* The discharge of capacitor in the primary circuit of an induction coil with a closed secondary circuit is oscillatory and governed by the equations

$$L\frac{dx}{dt}+M\frac{dy}{dt}+\frac{1}{C}\int x\,dt = f(t) \quad \text{and}$$
$$M\frac{dx}{dt}+N\frac{dy}{dt}=0,$$

where L, M, N, and C are all positive constants and $f(t)$ is a forcing function. Find the differential equation satisfied by the discharge $x(t)$, and show that when $LN-M^2$ is small and positive the complementary function for the discharge $x(t)$ exhibits rapid oscillations.

Background material

21.* Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the differential equation

$$a_0(x)y''+a_1(x)y'+a_2(x)y=0,$$

defined on some interval I. Then the **Abel formula** for the Wronskian is

$$W(y_1(x), y_2(x)) = W(y_1(x_0), y_2(x_0)) \times \exp\left(-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}dt\right),$$

where x_0 is any point in the interval I. Verify this result for the differential equation

$$x^2y''-2xy'-4y=0,$$

given that two linearly independent solutions over any interval that does not contain the origin are $1/x$ and x^4. Conclude that the choice of the point x_0 entering into the constant factor $W(y_1(x_0), y_2(x_0))$ is immaterial.

22.* Complete the details of the following outline proof of the Abel formula. Show that the derivative of the Wronskian of the functions in Exercise 21 can be written

$$W(y_1(x), y_2(x))' = y_1(x)y_2'' - y_2(x)y_1''(x).$$

Use the fact that $y_1(x)$ and $y_2(x)$ are solutions of the differential equation to show that

$$W' = -\frac{a_1(x)}{a_0(x)}W,$$

and by integrating over the interval $x_0 \le t \le x$ derive the result

$$W(y_1(x), y_2(x)) = W(y_1(x_0), y_2(x_0)) \times \exp\left(-\int_{x_0}^{x}\frac{a_1(t)}{a_0(t)}dt\right).$$

6.4 Undetermined Coefficients: Particular Integrals

Like the nonhomogeneous second order constant coefficient differential equation considered in Section 6.2, a **particular integral** $y_p(x)$ of the nonhomogeneous linear higher order constant coefficient differential equation

$$\boxed{\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy}{dx}+a_ny=f(x)} \tag{53}$$

is a solution of the equation that does not contain arbitrary constants, so

$$\frac{d^ny_p}{dx^n}+a_1\frac{d^{n-1}y_p}{dx^{n-1}}+\cdots+a_{n-1}\frac{dy_p}{dx}+a_ny_p = f(x).$$

particular integral, complementary function, and undetermined coefficients

The **complementary function** $y_c(x)$ associated with (53) is the general solution of the homogeneous form of the equation

$$\frac{d^n y_c}{dx^n} + a_1 \frac{d^{n-1} y_c}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy_c}{dx} + a_n y_c = 0,$$

considered in Section 6.3. It follows from the definitions of $y_c(x)$ and $y_p(x)$ and the linearity of the equation that the general solution $y(x)$ of (53) can be written

$$y(x) = y_c(x) + y_p(x). \tag{54}$$

A particular integral of (53) can be found by the **method of undetermined coefficients** whenever the nonhomogeneous term $f(x)$ is a linear combination of elementary functions such as polynomials, exponentials, and sine or cosine functions.

The method depends for its success on recognizing the general form of a function that when substituted into the left-hand side of (53) yields the general form of the nonhomogeneous term $f(x)$ on the right-hand side. Undetermined coefficients are involved because although the general form of a particular integral $y_p(x)$ can be guessed from the function $f(x)$, any multiplicative constants (the undetermined coefficients) involved will not be known. Their values are found by substituting the possible form for $y_p(x)$ into the left-hand side of (53) and equating the undetermined coefficients of terms on the left of the equation to the known coefficients of corresponding terms in $f(x)$ on the right. The approach is illustrated in the following example.

EXAMPLE 6.12

Find the general solution of

$$y'' + 5y' + 6y = 4e^{-x} + 5\sin x.$$

Solution The general solution is

$$y(x) = y_c(x) + y_p(x),$$

where $y_c(x)$ is the complementary function satisfying the homogeneous form of the equation

$$y_c'' + 5y_c' + 6y_c = 0,$$

and $y_p(x)$ is a particular integral that corresponds to the nonhomogeneous term $4e^{-x} + 5\sin x$.

The characteristic equation is

$$\lambda^2 + 5\lambda + 6 = 0,$$

with the roots $\lambda_1 = -2$ and $\lambda_2 = -3$ corresponding to the linearly independent solutions e^{-2x} and e^{-3x}, so the complementary function is

$$y_c(x) = C_1 e^{-2x} + C_2 e^{-3x},$$

where C_1 and C_2 are arbitrary constants.

To find a particular integral, we notice first that neither the term e^{-x} nor the term $\sin x$ is contained in the complementary function. This means that the only form of particular integral $y_p(x)$ that can produce the nonhomogeneous term $4e^{-x} + 5\sin x$ is

$$y_p(x) = Ae^{-x} + B\sin x + C\cos x,$$

undetermined coefficients

where A, B, and C are the *undetermined coefficients* that must be found.

Substituting this expression for $y_p(x)$ into the differential equation leads to the result

$$(Ae^{-x} - B\sin x - C\cos x) + 5(-Ae^{-x} + B\cos x - C\sin x) \\ + 6(Ae^{-x} + B\sin x + C\cos x) = 4e^{-x} + 5\sin x.$$

When we collect terms involving e^{-x}, $\sin x$, and $\cos x$ this becomes

$$2Ae^{-x} + 5(B - C)\sin x + 5(B + C)\cos x = 4e^{-x} + 5\sin x.$$

If $y_p(x)$ is a particular integral, this expression must be an identity (true for all x), but this is only possible if the coefficients of corresponding functions of x on either side of the equation are identical. Equating corresponding coefficients gives

$$\begin{aligned} &(\text{coefficients of } e^{-x}) && 2A = 4, \quad \text{so } A = 2 \\ &(\text{coefficient of } \sin x) && 5(B - C) = 5 \\ &(\text{coefficient of } \cos x) && 5(B + C) = 0. \end{aligned}$$

Solving the last two equations for B and C gives $B = 1/2$, $C = -1/2$, so the particular integral is

$$y_p(x) = 2e^{-x} + (1/2)\sin x - (1/2)\cos x.$$

Substituting $y_c(x)$ and $y_p(x)$ into $y(x) = y_c(x) + y_p(x)$ shows that the general solution is

$$y(x) = C_1e^{-2x} + C_2e^{-3x} + 2e^{-x} + (1/2)\sin x - (1/2)\cos x.$$

A complication arises if a term in the nonhomogeneous term $f(x)$ is contained in the complementary function, as illustrated in the next example.

EXAMPLE 6.13

Find a particular integral of the equation

$$y'' + y' - 12y = e^{3x}.$$

Solution This equation has the complementary function

$$y_c(x) = C_1e^{3x} + C_2e^{-4x},$$

so e^{3x} is contained in both the nonhomogeneous term and the complementary function.

An attempt to find a particular integral of the form $y_p(x) = Ae^{3x}$ will fail, because e^{3x} is a solution of the homogeneous form of the equation, so its substitution into the left-hand side of the differential equation will lead to the contradiction $0 = e^{3x}$. To overcome this difficulty we need to seek a more general particular integral that, when substituted into the differential equation, produces a multiple of e^{3x} whose scale factor can be equated to the coefficient of the nonhomogeneous

term and other terms that cancel. As exponentials are involved, a natural choice is $y_p(x) = Axe^{3x}$.

Differentiation of $y_p(x)$ gives

$$y_p'(x) = Ae^{3x} + 3Axe^{3x} \quad \text{and} \quad y_p''(x) = 6Ae^{3x} + 9Axe^{3x}.$$

Substituting these results into the differential equation gives

$$6Ae^{3x} + 9Axe^{3x} + Ae^{3x} + 3Axe^{3x} - 12Axe^{3x} = e^{3x},$$

so after cancellation of the terms in Axe^{3x} this reduces to

$$7Ae^{3x} = e^{3x},$$

showing that $A = 1/7$. So the required particular integral is

$$y_p(x) = \frac{1}{7}xe^{3x}.$$

Table 6.2 lists the form of particular integral that correspond to the most common nonhomogeneous terms. Each of its entries can be constructed by using arguments similar to the one just given. When the nonhomogeneous term is a linear combination of terms in the table, the form of $y_p(x)$ is found by adding the forms of the corresponding particular integrals.

EXAMPLE 6.14

Find the general solution of

$$y''' - 5y'' + 6y' = x^2 + \sin x.$$

some typical examples

Solution The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda = 0, \quad \text{or} \quad \lambda(\lambda^2 - 5\lambda + 6) = 0,$$

with the roots $\lambda_1 = 0$, $\lambda_2 = 2$, and $\lambda_3 = 3$, so the complementary function is

$$y_c(x) = C_1 + C_2e^{2x} + C_3e^{3x}.$$

The function x^2 on the right-hand side is not contained in the complementary function, but there is no undifferentiated term involving $y(x)$ in the equation, so from Step 2(b) in Table 6.2 the appropriate form of particular integral corresponding to this term is

$$Ax + Bx^2 + Cx^3.$$

The function $\sin x$ is not contained in the complementary function, so the form of particular integral appropriate to this term is seen from Step 4(a) to be

$$D\sin x + E\cos x.$$

Combining these two forms shows that the general form of $y_p(x)$ is

$$y_p(x) = Ax + Bx^2 + Cx^3 + D\sin x + E\cos x.$$

Substituting $y_p(x)$ into the differential equation gives

$$(6C - D\cos x + E\sin x) - 5(2B + 6Cx - D\sin x - E\cos x)$$
$$+ 6(A + 2Bx + 3Cx^2 + D\cos x - E\sin x) = x^2 + \sin x.$$

how to find a particular integral using undetermined coefficients

TABLE 6.2 Particular Integrals by the Method of Undetermined Coefficients

The method applies to the linear constant coefficient differential equation

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x),$$

which has the characteristic equation

$$\lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0,$$

with the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the complementary function

$$y_c(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x),$$

where $y_1(x), y_2(x), \ldots, y_n(x)$ are the linearly independent solutions of the homogeneous equation appropriate to the nature of the roots.

1. $f(x) = \text{constant}.$ $(\lambda \neq 0)$

Include in $y_p(x)$ the constant term K.

2. $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m.$

(a) If the left-hand side of the differential equation contains an undifferentiated term $y(x)$, include in $y_p(x)$ the polynomial

$$A_0 x^m + A_1 x^{m-1} + \cdots + A_m.$$

(b) If the left-hand side of the differential equation contains no undifferentiated function of $y(x)$, and the lowest order derivative is $d^s y/dx^s$, include in $y_p(x)$ the polynomial

$$A_0 x^{m+s} + A_1 x^{m+s-1} + \cdots + A_m x^s.$$

3. $f(x) = Pe^{ax}.$

(a) If e^{ax} is not contained in the complementary function, include in $y_p(x)$ the term

$$Be^{ax}.$$

(b) If the complementary function contains the terms $e^{ax}, xe^{ax}, \ldots, x^m e^{ax}$, include in $y_p(x)$ the term

$$Bx^{m+1} e^{ax}.$$

4. $f(x)$ contains terms in $\cos px$ and/or $\sin px$.

(a) If $\cos px$ and/or $\sin px$ are not contained in the complementary function, include in $y_p(x)$ the terms

$$P\cos px + Q\sin px.$$

(b) If the complementary function contains the terms $x\cos px$ and/or $x\sin px$, include in $y_p(x)$ terms of the form

$$x^2(P\cos px + Q\sin px).$$

(*continued*)

TABLE 6.2 (*continued*)

(c) If the complementary function contains the terms $x^2 \cos px$ and/or $x^2 \sin px$, include in $y_p(x)$ terms of the form

$$x^3(P\cos px + Q\sin px).$$

5. $f(x)$ contains terms in $e^{px}\cos qx$ and/or $e^{px}\sin qx$.

(a) If $e^{px}\cos qx$ and/or $e^{px}\sin qx$ are not contained in the complementary function, include in $y_p(x)$ terms of the form

$$e^{px}(R\cos qx + S\sin qx).$$

(b) If the complementary function contains $xe^{px}\cos qx$ and/or $xe^{px}\sin qx$, include in $y_p(x)$ terms of the form

$$x^2e^{px}(R\cos qx + S\sin qx).$$

6. The required particular integral $y_p(x)$ is the sum of all the terms produced by identifying each term belonging to $f(x)$ with one of the types of term listed above.

7. The values of the undetermined coefficients K, A_0, A_1, $\dots$, A_m, B, P, Q, R, and S are found by substituting $y_p(x)$ into the differential equation, equating the coefficients of corresponding functions on either side of the equation to make the result an identity, and then solving the resulting simultaneous equations for the undetermined coefficients.

Equating coefficients of corresponding functions on each side of this expression to make it an identity, we have

$$\begin{aligned}
&(\text{constant terms}) && 6C - 10B + 6A = 0,\\
&(\text{terms in } x) && -30C + 12B = 0,\\
&(\text{terms in } x^2) && 18C = 1,\\
&(\text{terms in} \sin x) && 5D - 5E = 1,\\
&(\text{terms in} \cos x) && 5D + 5E = 0.
\end{aligned}$$

Solving these simultaneous equations gives $A = 19/108$, $B = 5/36$, $C = 1/18$, $D = 1/10$, and $E = -1/10$, so the particular integral is

$$y_p(x) = \frac{19}{108}x + \frac{5}{36}x^2 + \frac{1}{18}x^3 + \frac{1}{10}\sin x - \frac{1}{10}\cos x.$$

Combining this with the complementary function shows the general solution to be

$$y(x) = C_1 + C_2e^{2x} + C_3e^{3x} + \frac{19}{108}x + \frac{5}{36}x^2 + \frac{1}{18}x^3 + \frac{1}{10}\sin x - \frac{1}{10}\cos x. \quad ■$$

The existence and uniqueness of solutions of initial value problems for nonhomogeneous linear differential equations are guaranteed by the following theorem, which is a direct extension of Theorem 6.3.

THEOREM 6.4

more on existence and uniqueness: this time for nonhomogeneous equations

Existence and uniqueness of solutions of nonhomogeneous linear equations Let the coefficients and nonhomogeneous term of differential equation (53) be continuous functions over an interval $a < x < b$ that contains the point x_0. Then a unique solution exists on this interval that satisfies the initial conditions

$$y(x_0) = k_0, \quad y^{(1)}(x_0) = k_1, \ldots, y^{(n-1)}(x_0) = k_{n-1}.$$

Proof As before, the proof of the existence of solutions of variable coefficient equations will be omitted, while the existence of solutions of constant coefficient equations has already been established. This only leaves the proof of uniqueness that follows along the same lines as those of Theorem 6.3, with $y(x)$ replaced by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n(x) + y_p(x),$$

and the system of equations determining $c_1, c_2, \ldots, c_n$ replaced by

$$\begin{aligned}
&c_1 y_1(x_0) + c_2 y_2(x_0) + \cdots + c_n(x) y_n(x_0) = k_0 - y_p(x_0) \\
&c_1 y_1^{(1)}(x_0) + c_2 y_2^{(1)}(x_0) + \cdots + c_n y_n^{(1)}(x_0) = k_1 - y_p'(x_0) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&c_1 y_1^{(n-1)}(x_0) + c_2 y_2^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = k_{n-1} - y_p^{(n-1)}(x_0).
\end{aligned}$$

The constants $c_1, c_2, \ldots, c_n$ are uniquely determined by this system because, as with Theorem 6.3, the determinant of the coefficients is the Wronskian and so is nonvanishing for $x = x_0$. ■

EXAMPLE 6.15

Solve the initial value problem

$$y'' + 4y' + 3y = e^{-x}, \quad \text{with } y(0) = 2, \quad y'(0) = 1.$$

Solution The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = 0,$$

with the roots $\lambda_1 = -1$ and $\lambda_2 = -3$, so the complementary function is

$$y_c(x) = C_1 e^{-x} + C_2 e^{-3x}.$$

The nonhomogeneous term e^{-x} is contained in the complementary function, so by Step 3(b) in Table 6.2 we must seek a particular integral of the form

$$y_p(x) = Axe^{-x}.$$

Substituting the expression for $y_p(x)$ into the differential equation gives

$$(-2Ae^{-x} + Axe^{-x}) + 4(Ae^{-x} - Axe^{-x}) + 3Axe^{-x} = e^{-x}, \quad \text{or} \quad 2Ae^{-x} = e^{-x},$$

showing that $A = 1/2$. So, in this case, the particular integral is $y_p(x) = (1/2)xe^{-x}$ and the general solution is

$$y(x) = C_1 e^{-x} + C_2 e^{-3x} + (1/2)xe^{-x}.$$

The initial condition $y(0) = 2$ will be satisfied if

$$2 = C_1 + C_2,$$

and the initial condition $y'(0) = 1$ will be satisfied if

$$1/2 = -C_1 - 3C_2,$$

so $C_1 = 13/4$ and $C_2 = -5/4$. Substituting these values for C_1 and C_2 in the general solution gives the solution of the initial value problem

$$y(x) = \left(\frac{13}{4} + \frac{1}{2}x\right)e^{-x} - \frac{5}{4}e^{-3x}.$$

Summary

The determination of particular integrals for nonhomogeneous equations is important, and the method of undetermined coefficients that was described in this section is the simplest method by which they can be found. The method is only applicable to nonhomogeneous terms formed by a sum of polynomials, exponentials, trigonometric functions, and certain of their combinations. It depends for its success on recognizing the general form of function that, when substituted into the left of the differential equation, produces terms of the type found in the nonhomogeneous term on the right. The method involves substituting a linear combination of such terms with arbitrary constant multipliers (the undetermined coefficients) into the left of the equation and matching the constants so the terms that result are identical to the terms on the right.

EXERCISES 6.4

Find the general solutions of the following differential equations.

1. $y'' + 2y' - 3y = 4 + x + 4e^{2x}$.
2. $y'' + 4y' + 4y = 2 - \sin 3x$.
3. $y'' + 2y' + y = 5 + x^2e^x$.
4. $y'' - 4y' + 4y = 3x^2 + 2e^{3x}$.
5. $y'' + 4y' + 4y = \sin x - 2\cos x$.
6. $y'' + 4y' + 5y = \sin x$.
7. $y'' + 2y' + 2y = 1 + x + e^{-x}$.
8. $y''' + 5y'' + 6y' = 3\sin x + 5x + x^2$.
9. $y''' + 2y'' + 2y' = 2 - 4x^2$.
10. $y'' + 2y' + 2y = \sin x$.
11. $y'' - 7y' + 12y = x + e^{2x} + e^{3x}$.
12. $y'' + 4y' + 5y = 3 + 2e^{-2x}$.
13. $y'' + 2y' - 8y = 3x\cos 4x$.
14. $y'' + 2y' - 15y = 3 + 2x\sin x$.
15. $y'' + 9y = 2\cos 3x + \sin 3x$.
16. $y'' - 4y = 3e^{2x} + 4e^{-2x}$.
17. $y'' + 3y' + 2y = x^2 + 3e^{-2x}$.
18. $y''' + y'' + 3y' - 5y = 4e^{-x}$.
19. $y'' + 4y' + 5y = e^{-2x}\sin x$.
20. $y'' + 4y' + 5y = x^2 - e^{-2x}\cos x$.

In Exercises 21 through 28 solve the initial value problems. Where the characteristic equation is of degree 3, at least one root is an integer and can be found by inspection.

21. $y'' + 6y' + 13y = e^{-3x}\cos x$, with $y(0) = 2$, $y'(0) = 1$.
22. $y'' - 4y' + 5y = e^{2x}\cos x$, with $y(0) = 0$, $y'(0) = 2$.
23. $y'' + 9y = 7 + 2\sin 3x - 4\cos 3x$, with $y(0) = -1$, $y'(0) = 1$.
24. $y'' + 4y' + 5y = x + \sin x$, with $y(0) = -1$, $y'(0) = 0$.
25. $y'' - 2y' + 5y = 1 + e^{-x}$, with $y(0) = 2$, $y'(0) = 1$.
26. $y'' + 4y' + 5y = 2 + e^{-2x}\sin x$, with $y(0) = 0$, $y'(0) = 0$.
27. $y''' + y'' - 2y = 3 + 2\cos x$, with $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$.
28. $y''' + y'' - y' - y = 2 + e^{-x}$, with $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$.

6.5 Cauchy–Euler Equation

Cauchy–Euler equation

One of the simplest linear variable coefficient differential equations is the homogeneous second order **Cauchy–Euler** equation, whose standard form is

$$x^2\frac{d^2y}{dx^2} + a_1x\frac{dy}{dx} + a_2y = 0. \tag{55}$$

The solution of this homogeneous equation can be reduced to a simple algebraic problem by seeking a solution of the form

$$y(x) = Ax^m, \tag{56}$$

where A is an arbitrary constant, and the permissible values of m are to be determined.

Differentiating $y(x)$ to obtain

$$\frac{dy}{dx} = mAx^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = m(m-1)Ax^{m-2} \tag{57}$$

and substituting these expressions into the Cauchy–Euler equation gives the following quadratic equation for m:

$$m(m-1) + a_1 m + a_2 = 0. \tag{58}$$

When this equation has two distinct real roots $m = \alpha$ and $m = \beta$, the general solution of (55) is

$$y(x) = C_1 x^\alpha + C_2 x^\beta, \tag{59}$$

but if the two roots are real and equal with $m = \mu$, the general solution of (55) is

$$y(x) = C_1 x^\mu + C_2 x^\mu \ln|x|, \tag{60}$$

where C_1 and C_2 are arbitrary real constants.

If the equation for m has the complex conjugate roots $m = \alpha \pm i\beta$, substitution confirms that the general solution of (55) is

$$y(x) = C_1 x^\alpha \cos(\beta \ln|x|) + C_2 x^\alpha \sin(\beta \ln|x|). \tag{61}$$

The second solution $x^\mu \ln|x|$ in (60) can be obtained from the method of Section 6.7 by using the known solution $y_1(x) = x^\mu$ to find a second linearly independent solution $y_2(x)$. The form of solution (61) follows from writing the general solution as $y(x) = A\exp(\alpha + i\beta) + B\exp(\alpha - i\beta)$, with A an arbitrary complex constant and B its complex conjugate so that $y(x)$ is real.

EXAMPLE 6.16 Find the general solution of

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 2y = 0 \quad \text{for } x \neq 0.$$

Solution The equation for m is

$$m(m-1) + 3m + 2 = 0,$$

with the roots $m = -1 \pm i$. The general solution is thus

$$y(x) = C_1 x^{-1}\cos(\ln|x|) + C_2 x^{-1}\sin(\ln|x|).$$ ■

Summary

The Cauchy–Euler equation is the simplest linear variable coefficient equation for which a closed form analytical solution can be found. The solution is obtained by recognizing that it must be of the form $y(x) = Ax^m$ and finding the permissible values of m.

EXERCISES 6.5

Find the general solutions of the following Cauchy–Euler equations.

1. $x^2y'' + 3xy' - 3y = 0.$

2. $x^2y'' + 3xy' + 5y = 0.$

3. $x^2y'' + 5xy' + 9y = 0.$

4. $x^2y'' - 3xy' - 5y = 0.$

5. $x^2y'' + 3xy' - 8y = 0.$

6. $x^2y'' + 2xy' + 4y = 0.$

7. $x^2y'' + 6xy' + 4y = 0.$

8. $x^2y'' + xy' + 4y = 0.$

9. $x^2y'' + 4xy' + 4y = 0.$

10. $x^2y'' + 3xy' + 6y = 0.$

11. With the change of variable $x = e^t$, we find using the chain rule that

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

Use these results to show that this change of variable transforms a Cauchy–Euler equation into a constant coefficient equation, and solve Exercise 3 by this method.

12. Use the substitution $y(x) = Ax^m$ to solve the third order Cauchy–Euler equation

$$x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$$

13. Use the substitution of Exercise 11 to solve the Cauchy–Euler equation in Exercise 12.

14. Express dy/dx, d^2y/dx^2, and d^3y/dx^3 in terms of dy/dt, d^2y/dt^2, and d^3y/dt^3 if $ax + b = e^t$. Use the substitution to show that the general solution of

$$(2x+3)^3y''' + 3(2x+3)y' - 6y = 0$$

is

$$y(x) = C_1(2x+3) + C_2(2x+3)^{1/2} + C_3(2x+3)^{3/2} \quad \text{for } x > 0.$$

6.6 Variation of Parameters and the Green's Function

Variation of Parameters

The method of **variation of parameters**, perhaps more properly called **variation of constants**, is a powerful method used to find a particular integral of a linear differential equation once its complementary function is known. In what follows the method will be developed for a general linear second order variable coefficient differential equation, though it is easily extended to include linear variable coefficient differential equations of any order.

As linear constant coefficient equations are a special case of variable coefficient equations, the method enables particular integrals to be found for all linear equations. The method also has the advantage that no special cases arise due to the nonhomogeneous term being included in the complementary function.

Consider the general linear second order differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x), \tag{62}$$

idea underlying the method of variation of parameters

defined on some interval $\alpha \le x \le \beta$ over which $a(x)$, $b(x)$, and $f(x)$ are defined and continuous. Let $y_1(x)$ and $y_2(x)$ be two known linearly independent solutions

of the homogeneous form of (62), so the complementary function is

$$y_c(x) = C_1 y_1(x) + C_2 y(x). \tag{63}$$

The idea underlying the method of variation of parameters, and from which it derives its name, is to replace the constants C_1 and C_2 by the unknown functions $u_1(x)$ and $u_2(x)$, and then to seek a particular integral of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{64}$$

Two equations are needed in order to determine $u_1(x)$ and $u_2(x)$, and the first of these is obtained as follows. Differentiation of (64) gives

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) + u_1'(x)y_1(x) + u_2'(x)y_2(x),$$

so by requiring $u_1(x)$ and $u_2(x)$ to be such that the last two terms vanish, we have

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x), \tag{65}$$

subject to the condition

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0. \tag{66}$$

Equation (66) is the *first* condition to be imposed on $u_1(x)$ and $u_2(x)$, and a second condition is obtained as follows. Differentiating (65) gives

$$y_p''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x), \tag{67}$$

so substituting (64), (65), and (67) into (62), followed by grouping terms, gives

$$\begin{aligned} u_1[y_1'' + a(x)y_1' + b(x)y_1] + u_2[y_2'' + a(x)y_2' + b(x)y_2] \; + \\ + \; u_1'y_1' + u_2'y_2' = f(x). \end{aligned} \tag{68}$$

As $y_1(x)$ and $y_2(x)$ are both solutions of differential equation (62) with $f(x) = 0$, the expressions multiplying $u_1(x)$ and $u_2(x)$ both vanish identically, reducing (68) to the *second* condition on $u_1(x)$ and $u_2(x)$,

$$u_1'y_1' + u_2'y_2' = f(x). \tag{69}$$

The functions $u_1(x)$ and $u_2(x)$ can now be found by solving equations (66) and (69). Solving these for $u_1'(x)$ and $u_2'(x)$ gives

$$u_1'(x) = \frac{-y_2(x)f(x)}{W(x)} \quad \text{and} \quad u_2'(x) = \frac{y_1(x)f(x)}{W(x)}, \tag{70}$$

where

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \tag{71}$$

is the Wronskian of $y_1(x)$ and $y_2(x)$ and so is never zero.

After integration, results (70) become

$$u_1(x) = -\int \frac{y_2(x)\,f(x)}{W(x)}dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)\,f(x)}{W(x)}dx. \tag{72}$$

the general solution

Finally, combining (64) and (72), we find that

$$y(x) = -y_1(x)\int \frac{y_2(x)\,f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)\,f(x)}{W(x)}dx. \tag{73}$$

This result represents the general solution of (62), because each indefinite integral has associated with it an additive arbitrary constant, and if these are $-C_1$ and C_2, say, they include in $y(x)$ the complementary function $y_c(x) = C_1y_1(x) + C_2y_2(x)$. When these constants are set equal to zero result (73) reduces to the particular integral $y_p(x)$.

Rule for the method of variation of parameters

how to apply the method of variation of parameters

1. Write the differential equation in the standard form

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x).$$

2. Find two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the homogeneous form of the differential equation and construct the equations

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \quad \text{and} \quad u_1'y_1' + u_2'y_2' = f(x).$$

3. Solve the equations in Step 2 for $u_1'(x)$ and $u_2'(x)$ and integrate to find $u_1(x)$ and $u_2(x)$, each with an arbitrary additive constant of integration.
4. The general solution of the differential equation is then given by

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x).$$

Or, alternatively, after finding $y_1(x)$ and $y_2(x)$:

5. Substitute into

$$y(x) = -y_1(x)\int \frac{y_2(x)\,f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x)\,f(x)}{W(x)}dx,$$

where

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

6. The result of Step 5 becomes the particular integral $y_p(x)$ if the arbitrary integration constants are set equal to zero.

The example that follows shows how the method of variation of parameters deals automatically with the presence of a nonhomogeneous term in the complementary function of a constant coefficient equation.

EXAMPLE 6.17

a simple example that could also be solved by undetermined coefficients

Find the general solution of the second order differential equation

$$y'' + 2y' + y = xe^{-x}.$$

Solution The characteristic equation is

$$\lambda^2 + 2\lambda + 1 = 0,$$

with the repeated root $\lambda = -1$. Thus, the complementary function is

$$y_c(x) = C_1 e^{-x} + C_2 x e^{-x}.$$

Two linearly independent solutions are thus

$$y_1(x) = e^{-x} \quad \text{and} \quad y_2(x) = xe^{-x},$$

while the nonhomogeneous term is $f(x) = xe^{-x}$. The Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{-x}(e^{-x} - xe^{-x}) + e^{-x}xe^{-x} = e^{-2x},$$

so substituting in (73) shows that the particular integral is

$$y_p(x) = -e^{-x}\int x^2 \mathrm{d}x + xe^{-x}\int x \mathrm{d}x = \frac{1}{6}x^3 e^{-x}.$$

The general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + \frac{1}{6}x^3 e^{-x}.$$

This result could, of course, have been found by the method of undetermined coefficients. ■

The next example shows how the method of variation of parameters determines a particular integral for a constant coefficient equation whose particular integral could not have been found by using undetermined coefficients.

EXAMPLE 6.18

an example that could not be solved by undetermined coefficients

Find the general solution of the differential equation

$$y'' + y = \csc x$$

in any interval in which $x \neq n\pi$, for $n = 1, 2, \ldots$.

Solution It follows at once that the complementary function is

$$y_c(x) = C_1 \cos x + C_2 \sin x,$$

so two linearly independent solutions are

$$y_1(x) = \cos x \quad \text{and} \quad y_2(x) = \sin x.$$

The Wronskian $W(x) = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$, and $f(x) = 1/\sin x$, so substituting into (73) shows that the particular integral is

$$y_p(x) = -\cos x \int \mathrm{d}x + \sin x \int \cot x \mathrm{d}x.$$

As $\int \cot x\, dx = \ln|\sin x|$,

$$y_p(x) = -x\cos x + \sin x \ln|\sin x|,$$

and the general solution is

$$y(x) = C_1\cos x + C_2\sin x - x\cos x + \sin x \ln|\sin x|,$$

in any interval in which $x \neq n\pi$, for $n = 1, 2, \ldots$, because $\ln|\sin n\pi| = \infty$. Although this is a constant coefficient equation, it is unlikely that its particular integral could have been found by the method of undetermined coefficients. ■

The last example shows how the method of variation of parameters determines a particular integral for a linear second order variable coefficient equation.

EXAMPLE 6.19 Find the general solution of the second order variable coefficient equation

$$x^2y'' - 3xy' + 4y = \ln x \quad (x > 0).$$

application to a variable coefficient equation

Solution This is a Cauchy–Euler equation, and the method of Section 6.5 shows that its complementary function is

$$y_c(x) = C_1x^2 + C_2x^2\ln x, \quad \text{for} \quad x > 0,$$

so two linearly independent solutions are

$$y_1(x) = x^2 \quad \text{and} \quad y_2(x) = x^2\ln x \quad \text{for} \quad x > 0.$$

A routine calculation shows the Wronskian $W(x) = x^3$. Before identifying $f(x)$ the equation must be written in the standard form with the coefficient of y'' equal to 1. Dividing the differential equation by x^2 to bring it into the standard form shows that $f(x) = (\ln x)/x^2$.

Substitution into (73) then gives

$$y_p(x) = -x^2\int \frac{(\ln x)^2}{x^3}dx + x^2\ln\, x\int \frac{\ln x}{x^3}dx.$$

Integration by parts shows that

$$\int \frac{(\ln x)^2}{x^3} = -\frac{1}{2}\frac{(\ln x)^2}{x^2} - \frac{1}{2}\frac{\ln x}{x^2} - \frac{1}{4x^2} \quad \text{and} \quad \int \frac{\ln x}{x^3}dx = -\frac{1}{2}\frac{\ln x}{x^2} - \frac{1}{4x^2},$$

so using these results in the expression for $y_p(x)$ gives

$$y_p(x) = \frac{1}{4} + \frac{1}{4}\ln x \quad (x > 0).$$

The general solution is thus

$$y(x) = C_1x^2 + C_2x^2\ln x + \frac{1}{4} + \frac{1}{4}\ln x \quad (x > 0).$$

Although the complementary function of a Cauchy–Euler equation is easily determined, a particular integral is usually sufficiently complicated that its general form cannot be guessed and so must be found by the method of variation of parameters. ■

what happens if an integral has no known antiderivative

Finally, we remark that an application of the method of variation of parameters to the equation

$$y'' + y = (1 + x^2)^{1/2}$$

gives a particular integral in the form

$$y_p(x) = -\cos x \int (\sin x)(1 + x^2)^{1/2} \mathrm{d}x + \sin x \int (\cos x)(1 + x^2)^{1/2} \mathrm{d}x.$$

Neither of the two integrals involved can be evaluated in terms of known functions, so if an analytical solution is needed it must be obtained in series form. The Maclaurin series for the functions $(\sin x)(1 + x^2)^{1/2}$ and $(\cos x)(1 + x^2)^{1/2}$ are

$$(\sin x)(1 + x^2)^{1/2} = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \cdots \quad \text{and}$$
$$(\cos x)(1 + x^2)^{1/2} = x - \frac{1}{3}x^4 - \frac{13}{90}x^6 + \cdots.$$

Integrating these results and substituting in the expression for $y_p(x)$ gives

$$y_p(x) = -(\cos x)\left(\frac{1}{2}x^2 + \frac{1}{12}x^4 - \frac{1}{30}x^6 + \cdots\right) + \sin x\left(x - \frac{1}{15}x^5 + \frac{13}{630}x^7 + \cdots\right).$$

■

Let $y(x)$ satisfy the differential equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x), \tag{74}$$

defined on an interval $\alpha \le x \le \beta$, and let a be any point inside this interval. Then the general solution of (74) given in (73) can be put into a convenient form for solving the initial value problem for (74) when the initial conditions are $y(a) = 0$ and $y'(a) = 0$.

We start from the general solution in (73)

$$y(x) = -y_1(x)\int \frac{y_2(x) f(x)}{W(x)}dx + y_2(x)\int \frac{y_1(x) f(x)}{W(x)}dx. \tag{75}$$

Next, we rewrite the indefinite integral $\int \frac{y_2(x) f(x)}{W(x)}dx$ as the definite integral with a variable upper limit $\int_a^x \frac{y_2(t) f(t)}{W(t)}dt$ and an arbitrary fixed lower limit $x = a$. In this result, the additive arbitrary integration constant associated with the indefinite integral has been replaced by the arbitrary constant a in the lower integration limit. The implications of the lower limit will become apparent when an initial value problem is considered. A corresponding result holds for the second indefinite integral in (75). Using these results, taking the functions $y_1(x)$ and $y_2(x)$ under the respective integral signs as they are not involved in the integrations, and combining the integrals allows the general solution $y(x)$ to be written in the form

$$y(x) = \int_a^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t)dt. \tag{76}$$

Setting $x = a$ in this result shows that $y(a) = 0$. Differentiation of (76) with respect to x using Leibniz's rule

$$\frac{d}{dx}\int_{p(x)}^{q(x)} g(x,t)dt = \frac{dq}{dx}g(x,q) - \frac{dp}{dx}g(x,p) + \int_{p(x)}^{q(x)} \frac{\partial}{\partial x}g(x,t)dt$$

gives

$$y'(x) = \frac{y_1(x)y_2(x) - y_1(x)y_2(x)}{W(x)}f(x) + \int_a^x \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)}dt.$$

The first term on the right vanishes, and setting $x = a$ causes the integral to vanish, so we have shown that $y'(a) = 0$. Consequently, the integral

variation of parameters and initial value problems

$$y(x) = \int_a^x \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} f(t)dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x), \quad \text{with} \quad y(a) = y'(a) = 0.$$

EXAMPLE 6.20

Use result (76) to solve the initial value problem

$$y'' + 4y = 1 + \cos 2x, \quad \text{with } y(0) = y'(0) = 0.$$

Solution Two linearly independent solutions of the homogeneous equation are $y_1(x) = \sin 2x$ and $y_2(x) = \cos 2x$, so $W(t) = -2(\sin^2 2t + \cos^2 2t) = -2$. Substituting into (76) with $f(t) = 1 + \cos 2t$ gives

$$y(x) = \int_0^x \tfrac{1}{2}(\sin 2x\cos 2t - \sin 2t\cos 2x)(1 + \cos 2t)dt,$$

and so

$$y(x) = \tfrac{1}{4}(1 - \cos 2x + x\sin 2x).$$ ■

The Green's Function

An important result that can be derived from the general solution of (74) when expressed in the form given in (75) is obtained by considering a boundary value problem for the equation written in the standard form

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x), \tag{77}$$

and defined over the interval $a \le x \le b$.

Evaluating the first integral in (75) over the interval $b \le t \le x$, changing the sign by reversing the limits of integration, and then evaluating the second integral

over the interval $a \le t \le x$ gives

$$y(x) = y_2(x)\int_a^x \frac{y_1(t)}{W(t)} f(t)dt + y_1(x)\int_x^b \frac{y_2(t)}{W(t)} f(t)dt. \tag{78}$$

As $y_2(x)$ is not involved in the first integral, and $y_1(x)$ is not involved in the second integral, they may be taken under the respective integral signs so that (78) becomes

$$y(x) = \int_a^x \frac{y_1(t)y_2(x)}{W(t)} f(t)dt + \int_x^b \frac{y_1(x)y_2(t)}{W(t)} f(t)dt. \tag{79}$$

This can be written

$$y(x) = \int_a^b G(x,t) f(t)dt, \tag{80}$$

the Green's function

where the function $G(x,t)$ is called the **Green's function** for differential equation (77) defined over the interval $a \le x \le b$ and is defined as

$$G(x,t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{W(t)}, & a \le t \le x \\ \dfrac{y_1(x)y_2(t)}{W(t)}, & x \le t \le b. \end{cases} \tag{81}$$

Inspection of (81) shows $G(x,t)$ to be a continuous function of x for $a \le x \le b$. Differentiation of $G(x,t)$ with respect to x gives

$$G_x(x,t) = \begin{cases} \dfrac{y_1(t)y_2'(x)}{W(t)}, & a \le t \le x \\ \dfrac{y_1'(x)y_2(t)}{W(t)}, & x \le t \le b. \end{cases} \tag{82}$$

Examination of (82) shows that as t increases across $t = x$, the function $G_x(x,t)$ is discontinuous and experiences the jump

$$G_x(x,x_+) - G_x(x,x_-) = \frac{y_1'(x)y_2(x) - y_1(x)y_2'(x)}{W(x)} = -\frac{W(x)}{W(x)} = -1,$$

where x_+ is the limit at t decreases to x and x_- is the limit as t increases to x.

Now let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous differential equation, with $y_1(x)$ such that at $x = a$ it satisfies the homogeneous boundary condition

$$k_1 y_1(a) + K_1 y_1'(a) = 0,$$

and $y_2(x)$ such that at $x = b$ it satisfies the homogeneous boundary condition

$$k_2 y_2(b) + K_2 y_2'(b) = 0.$$

Then $G(x,t)$ is seen to satisfy these same homogeneous boundary conditions, and differentiation of (80) with respect to x, again using Leibniz's rule, shows that

the solution $y(x)$ also satisfies these homogeneous boundary conditions. Combining results shows that

$$y(x) = \int_a^b G(x,t)f(t)dt \quad \text{with} \quad G(x,t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{W(t)}, & a \le t \le x \\ \dfrac{y_1(x)y_2(t)}{W(t)}, & x \le t \le b \end{cases} \tag{83}$$

is the solution of the boundary value problem for the nonhomogeneous linear second order equation

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = f(x),$$

subject to the homogeneous boundary conditions

$$k_1y(a) + K_1y'(a) = 0 \quad \text{with} \quad k_2y(b) + K_2y'(b) = 0.$$

When using this approach, unless the Green's function itself is required, it is usually more convenient to obtain the solution directly from result (78). The advantage of the Green's function is that it characterizes all the essential features of the differential equation without reference to the nonhomogeneous term $f(x)$, so that once it is known (80) solves the homogeneous boundary value problem for any function $f(x)$.

fundamental properties of the Green's function

Properties of the Green's function defined over the interval $a \le x \le b$

Consider the boundary value problem

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0,$$

subject to the boundary conditions

$$k_1y(a) + K_1y'(a) = 0 \quad \text{and} \quad k_2y(b) + K_2y'(b) = 0$$

The Green's function in (81) has the following properties:

1. The piecewise defined Green's function $G(x,t)$ satisfies the differential equation in the respective intervals $a \le x < t$ and $t < x \le b$.
2. $G(x,t)$ is a continuous function of x for $a \le x \le b$.
3. $G(x,t)$ satisfies the homogeneous boundary conditions.
4. The function $G_x(x,t)$ is continuous for $a \le x < t$ and $t < x \le b$, but it is discontinuous across $x = t$ where it experiences the jump

$$G_x(x, x_+) - G_x(x, x_-) = -1.$$

EXAMPLE 6.21 Find the Green's function for the differential equation

$$x^2y'' - 2xy' + 2y = 3x^2$$

and use it to solve the boundary value problem when $y(1) = 0$ and $y'(2) = 0$.

Solution The homogeneous form of the equation is a Cauchy–Euler equation, and the method of Section 6.5 shows that it has the two linearly independent solutions $y_1(x) = x$ and $y_2(x) = x^2$, so the general solution is $y(x) = ax + bx^2$.

For the solution $y_1(x)$ we must use the form of this solution that satisfies the left boundary condition $y(1) = 0$, and this is easily seen to be $y_1(x) = x - x^2$. For the linearly independent solution $y_2(x)$ we must use the form of solution $y(x) = ax + bx^2$ that satisfies the right boundary condition $y'(2) = 0$. As $y'(x) = a + 2bx$, the condition $y'(2) = 0$ shows that $y_2(x) = 4x - x^2$. Using these results the Wronskian becomes $W(t) = 3t^2$.

The Green's function for this differential equation defined by (81) is

$$G(x,t) = \begin{cases} \dfrac{(t-t^2)(4x-x^2)}{3t^2}, & 1 \le t < x \\ \dfrac{(4t-t^2)(x-x^2)}{3t^2}, & x < t \le 2. \end{cases}$$

To find the function $f(x)$ we must write the equation in the standard form where the coefficient of y'' is 1, so

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 3,$$

showing that $f(x) = 3$. It now follows from (78), or from (80), that

$$y(x) = (4x - x^2)\int_1^x \frac{(t-t^2)}{3t^2}3dt + (x - x^2)\int_x^2 \frac{(4t-t^2)3}{3t^2}dt,$$

and so

$$y(x) = x^2(3\ln x - 2 - 4\ln 2) + 2x(1 + 2\ln 2).$$

It is easily checked that this is the required solution, because $y(1) = 0$, $y'(2) = 0$, and $y(x)$ satisfies the differential equation. ■

More information and examples relating to the material in Sections 6.1 to 6.6 can be found in any one of the references [3.3], [3.4], [3.15], and [3.16].

Summary

This section described the powerful method of variation of parameters that enables the general solution of a linear nonhomogeneous equation to be found from the linearly independent solutions (the basis functions) that enter into its complementary function. It takes automatic account of nonhomogeneous terms that contain one or more basis functions, and it enables particular integrals, and hence general solutions, to be found where the method of undetermined coefficients fails. It was shown how the general solution obtained by the method of variation of parameters can be rewritten in terms of a Green's function that characterizes all of the essential features of the differential equation without reference to the nonhomogeneous term. Knowledge of the Green's function enables a homogeneous boundary value problem to be solved for any given nonhomogeneous term on the right of the equation.

EXERCISES 6.6

In Exercises 1 through 13 find the general solution.

1. $y'' + y' - 2y = xe^x$.

2. $y'' - 5y' + 6y = x^2e^{3x}$.

3. $y'' + 5y' + 6y = x^2e^{-2x}$.

4. $y'' + 4y' + 4y = x\sin x$.

5. $y'' - 2y + y = 2e^x/x$.

6. $y'' + 4y' + 5y = e^{-2x}\sin x$.

7. $y'' + 4y' + 5y = xe^{-2x}\cos x$.

8. $y'' - 4y' + 4y = e^{2x}/x$.

9. $y'' + 16y = x^2e^x$.

10. $y'' + 16y = \sec x$.

11. $y'' + 3y' + 2y = 3/(1 + e^x)$.

12. $y'' + y = \tan x$.

13. $y'' + y = \sec^2 x$.

In Exercises 14 through 18 verify that the functions $y_1(x)$ and $y_2(x)$ are linearly independent solutions of homogeneous form of the stated differential equation, and use them to find a particular integral and a general solution of the given equation.

14. $x^2y'' - 4xy' + 6y = 2x + \ln x$, where $y_1(x) = x^2$ and $y_2(x) = x^3$.

15. $x^2y'' + 3xy' - 3y = \sqrt{x}$, where $y_1(x) = x$ and $y_2(x) = x^{-3}$.

16. $x^2y'' + 3xy' - 8y = 2\ln x$, where $y_1(x) = x^2$ and $y_2(x) = x^{-4}$.

17. $(1 - x^2)y'' - xy' + 4y = x$, where $y_1(x) = 2x^2 - 1$ and $y_2(x) = x(x^2 - 1)^{1/2}$.

18. $(1 - x^2)y'' - 2y' = 1$, where $y_1(x) = 1$ and $y_2(x) = x + 2\ln(x - 1)$.

In Exercises 19 through 22 use result (76) to solve the stated initial value problem.

19. $x^2y'' - 3xy' + 3y = 2x^2\ln x$, with $y(1) = 0$ and $y'(1) = 0$.

20. $y'' + 5y' + 6y = xe^{-2x}$, with $y(1) = 0$ and $y'(1) = 0$.

21. $y'' + y = 2\sec^2 x$, with $y(0) = 0$ and $y'(0) = 0$.

22. $y'' + 4y' + 5y = x$, with $y(0) = 0$ and $y'(0) = 0$.

In Exercises 23 through 26 find the Green's function for the given differential equation, subject to the associated homogeneous boundary conditions.

23. $y'' = f(x)$, with $y(0) = 0$ and $y(1) = 0$.

24. $y'' = f(x)$, with $y(0) = 0$ and $y'(1) = 0$.

25. $y'' + \lambda^2 y = f(x)$, with $y(0) = 0$ and $y(1) = 0$.

26. $y'' + \lambda^2 y = f(x)$, with $y(0) = 0$ and $y'(1) = 0$.

In Exercises 27 through 30 solve the given boundary value problem by means of a suitable Green's function.

27. $x^2y'' + xy' - y = x^2e^{-x}$, with $y(1) = 0$ and $y(2) = 0$.

28. $x^2y'' + 2xy' - 2y = x^3$, with $y(1) = 0$ and $y(2) = 0$.

29. $x^2y'' - 3xy' + 3y = x^2\ln x$, with $y'(1) = 0$ and $y(2) = 0$.

30. $x^2y'' - 3xy' = x^2$, with $y(1) = 0$ and $y(2) = 0$.

6.7 Finding a Second Linearly Independent Solution from a Known Solution: The Reduction of Order Method

reduction of order method

In working with homogeneous linear second order variable coefficient equations, it can happen that one solution $y_1(x)$ is known and it is necessary to find a second linearly independent solution $y_2(x)$. The method we now describe, called the **reduction of order** method, involves seeking a second solution of the form

$$y_2(x) = u(x)y_1(x), \tag{84}$$

where the function $u(x)$ is to be determined. Provided $u(x)$ is not constant, the solutions $y_1(x)$ and $y_2(x)$ will be linearly independent, because $y_1(x)$ and $y_2(x)$ will not be proportional.

The method will be developed using the homogeneous second order variable coefficient equation in the standard form

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0. \tag{85}$$

Differentiating (84) gives

$$\frac{dy_2}{dx} = \frac{du}{dx}y_1 + u\frac{dy_1}{dx}, \quad \text{and} \quad \frac{d^2y_2}{dx^2} = \frac{d^2u}{dx^2}y_1 + 2\frac{du}{dx}\frac{dy_1}{dx} + u\frac{d^2y_1}{dx^2}. \tag{86}$$

Substituting (84) and (86) into (85) and grouping terms gives

$$y_1u'' + (2y_1' + ay_1)u' + (y_1'' + ay_1' + by_1)u = 0. \tag{87}$$

As $y_1(x)$ is a solution of (85), the factor $y_1'' + ay_1' + by_1$ multiplying u is zero, causing the equation to be reduced to

$$\frac{d^2u}{dx^2} = -\left(\frac{2y_1'}{y_1} + a(x)\right)\frac{du}{dx}. \tag{88}$$

The substitution $v = du/dx$ reduces (88) to the first order variables separable equation

$$\frac{dv}{dx} = -\left(\frac{2y_1'}{y_1} + a(x)\right)v, \tag{89}$$

and it is from this reduction of order of the differential equation that the method derives its name.

Separating variables and integrating (89) we find that

$$\int \frac{dv}{v} = -\int \left(\frac{2y_1'}{y_1} + a(x)\right)dx + \ln C,$$

or

$$\ln(v/C) = -\int \left(\frac{2y_1'}{y_1} + a(x)\right)dx,$$

so

$$v(x) = C\frac{\exp\{-\int a(x)dx\}}{y_1^2}. \tag{90}$$

As $v = du/dx$, integration of (90) gives

$$u(x) = C\int \left[\frac{\exp\{-\int a(x)dx\}}{y_1^2}\right]dx + D,$$

where D is another arbitrary constant.

The arbitrary constant D can be set equal to zero, because when $u(x)$ is substituted in (84) the constant D will simply scale the solution $y_1(x)$. Furthermore, as any constant C that scales $u(x)$ will scale each term in the differential equation, its value is immaterial, so for convenience we set $C = 1$. Thus, the expression for $u(x)$ is given by

$$u(x) = \int \left[\frac{\exp\{-\int a(x)dx\}}{y_1^2}\right]dx. \tag{91}$$

Using this expression for $u(x)$ in (84) shows that the second linearly independent solution is

$$y_2(x) = y_1(x) \int \left[\frac{\exp\{-\int a(x)dx\}}{y_1^2} \right] dx. \tag{92}$$

Thus, in terms of $y_1(x)$, the general solution of (85) can be written

$$y(x) = C_1 y_1(x) + C_2 y_1(x) \int \left[\frac{\exp\{-\int a(x)dx\}}{y_1^2} \right] dx, \tag{93}$$

where C_1 and C_2 are arbitrary constants.

EXAMPLE 6.22

Given that $y_1(x) = e^{-3x}$ is a solution of $y'' + 6y' + 9y = 0$, find a second linearly independent solution, and hence find the general solution.

Solution The equation is in standard form with $a(x) = 6$ and $y_1(x) = e^{-3x}$, so

$$u(x) = \int \left(\frac{\exp\{-\int 6\,dx\}}{\exp(-6x)} \right) dx = \int dx = x,$$

showing that

$$y_2(x) = xe^{-3x}.$$

This result is to be expected, because the linear constant coefficient equation corresponds to case (III) with $\mu = -3$. The general solution is thus

$$y(x) = (C_1 + C_2 x)e^{-3x}.$$

■

EXAMPLE 6.23

Given that $y_1(x) = x^2$ is a solution of $x^2 y'' - 3xy' + 4y = 0$ for $x > 0$, find a second linearly independent solution, and hence find the general solution.

Solution Writing the equation in standard form (85) shows that $a(x) = -3/x$, so

$$\begin{aligned} u(x) &= \int \frac{\exp\{-\int \{-3/x)dx\}}{x^4} dx = \int \frac{\exp\{\ln x^3\}}{x^4} dx \\ &= \int \frac{dx}{x} = \ln x, \end{aligned}$$

from which it follows that the second linearly independent solution is

$$y_2(x) = x^2 \ln x \quad \text{for } x > 0.$$

The general solution is

$$y(x) = x^2(C_1 + C_2 \ln x).$$

■

The reduction of order method can lead to an expression for $u(x)$ that cannot be integrated analytically. In such cases, in order to find an analytical approximation to $y_2(x)$, the integrand in (92) must be expanded in powers of x and integrated term by term. This approach will be used in Chapter 8 in connection with series solutions of second order variable coefficient linear differential equations. See references [3.3] and [3.4].

Summary

It is often the case that one solution of a linear second order variable-coefficient homogeneous variable-coefficient equation can be found, often by inspection, though a second linearly independent solution cannot be found in similar fashion. This section showed how a known solution can be used to find a second linearly independent solution. It was shown that the second linearly independent solution of the original second order equation is determined in terms of a first order equation, and it is this feature that has caused this approach to be called the reduction of order method.

EXERCISES 6.7

In the following exercises, verify that $y_1(x)$ is a solution of the given differential equation and use it to find a second linearly independent solution.

1. $y'' - 5y' - 14y = 0$ with $y_1(x) = e^{7x}$.
2. $y'' + 4y = 0$, with $y_1(x) = \sin 2x$.
3. $y'' + 4y' + 5y = 0$, with $y_1(x) = e^{-x}\cos x$.
4. $x^2y'' + 3xy' + y = 0$, with $y_1(x) = 1/x$.
5. $x^2y'' - xy' + y = 0$, with $y_1(x) = x$.
6. $x^2y'' + xy' + y = 0$, with $y_1(x) = \cos(\ln x)$.
7. $xy'' + 2y' + xy = 0$, with $y_1(x) = \sin x/x$.
8. $x^2y'' + xy' + (x^2 - 1/4)y = 0$, with $y_1(x) = \sin x/\sqrt{x}$.
9. $x^2(\ln x - 1)y'' - xy' + y = 0$, with $y_1(x) = x$.
10. $(1 - x\cot x)y'' - xy' + y = 0$, with $y_1(x) = x$.

(Hint: When finding $\int -a(x)dx$, make the substitution $u = \sin x - x\cos$, and in the final integral make the substitution $v = \sin x/x$.)

6.8 Reduction to the Standard Form $u'' + f(x)u = 0$

When studying the properties of second order variable coefficient equations it is sometimes advantageous to reduce the equation

$$y'' + a(x)y' + b(x)y = 0 \tag{94}$$

the standard form of a linear variable coefficient equation

to the *standard form* for a second order equation

$$u'' + f(x)u = 0, \tag{95}$$

from which the first derivative term u' is missing. This reduction has many uses, one of which occurs in Section 8.6 when we derive the analytical form of Bessel functions of fractional order.

To accomplish the reduction we seek a solution of (94) of the form

$$y(x) = u(x)v(x), \tag{96}$$

and then try to choose $v(x)$ so the first derivative term in u vanishes. Differentiation of $y = uv$ gives $y' = uv' + u'v$ and $y'' = u''v + 2u'v' + uv''$, so substitution into equation (94) gives

$$u''v + (2v' + av)u' + (v'' + av' + bv)u = 0. \tag{97}$$

This result shows that the first derivative term u' will vanish if $v(x)$ is such that

$$2v' + av = 0, \tag{98}$$

which has the solution

$$v(x) = \exp\left[-\frac{1}{2}\int a(x)dx\right]. \tag{99}$$

From (98) we have $v' = -(1/2)av$ and $v'' = -(1/2)(a'v + av')$, so eliminating v' and v'' from (97) gives

$$u'' + \left[-\frac{1}{2}a'(x) - \frac{1}{4}a^2(x) + b(x)\right]u = 0. \tag{100}$$

Because of its importance, we record this result in the form of a theorem.

THEOREM 6.5

how to perform the reduction

Reduction to the standard form $u'' + f(x)u = 0$ The substitution $y(x) = u(x)v(x)$, with

$$v(x) = \exp\left[-\frac{1}{2}\int a(x)dx\right],$$

reduces the differential equation

$$y'' + a(x)y' + b(x)y = 0$$

to

$$u'' + f(x)u = 0,$$

where

$$f(x) = -\frac{1}{2}a'(x) - \frac{1}{4}a^2(x) + b(x).$$

■

EXAMPLE 6.24

Reduce the equation

$$4x^2y'' + 4xy' + (16x^2 - 1)y = 0$$

to standard form and hence find the general solution.

Solution Dividing the differential equation by $4x^2$ to reduce it to the form given in (94) shows that $a(x) = 1/x$ and $b(x) = 4 - 1/(4x^2)$. Applying the result of Theorem 6.5 then shows that

$$v(x) = \exp\left[-\frac{1}{2}\int(1/x)dx\right] = x^{-1/2} \quad \text{and} \quad f(x) = 4.$$

The equation for $u(x)$ is thus $u'' + 4u = 0$ with the general solution

$$u(x) = C_1\cos 2x + C_2\sin 2x,$$

but $y(x) = u(x)v(x) = x^{-1/2}u(x)$, so the general solution is

$$y(x) = C_1\sqrt{\frac{1}{x}}\cos 2x + C_2\sqrt{\frac{1}{x}}\sin 2x. \quad \blacksquare$$

See references [3.3] and [3.4].

Summary

The study of the properties of some homogeneous linear variable coefficient equations of the form $y'' + a(x)y' + b(x)y = 0$ is simplified if a change of variable can be found that reduces them to an equivalent form $u'' + f(x)u = 0$. This section showed how such a change of variable can be found, and used it to solve a variable coefficient equation for which the two linearly independent functions entering into its general solution are by no means obvious.

EXERCISES 6.8

Reduce the equations in Exercises 1 and 2 to standard form, but do not attempt to find their general solutions.

1. $x^2y'' - xy' + 9xy = 0$.

2. $x^2y'' + xy' + (x^2 - 9)y = 0$.

In Exercises 3 through 7 reduce the equation to standard form and hence find its general solution.

3. $y'' - 2y' + y = 0$.

4. $y'' + 4y' + 3y = 0$.

5. $y'' - 4y' + 5y = 0$.

6. $x^2y'' + xy' + (36x^2 - 1)y = 0$.

7. $xy'' + 2y' + xy = 0$.

6.9 Systems of Ordinary Differential Equations: An Introduction

Physical problems that give rise to ordinary differential equations often do so in the form of coupled systems of first order linear differential equations, or systems of second order equations that are more easily treated if reduced to a first order system. A very simple example of this type was encountered in Section 5.2(d), where two first order equations were derived that linked the current i and the charge q flowing in an R–L–C circuit at time t. In that case it was convenient to eliminate the current i to obtain a simple second order equation for the current q that could be solved by the methods of Section 6.1 and 6.2.

Another example is the three-loop electric circuit shown in Fig. 6.12. In the circuit H is an inductance; C_1 and C_2 are capacitances; R_1, R_2, and R_3 are resistors; V_0 is an applied voltage; i_1, i_2, and i_3 are circulating currents; and q_2 and q_3 are the charges on the respective capacitances C_1 and C_2.

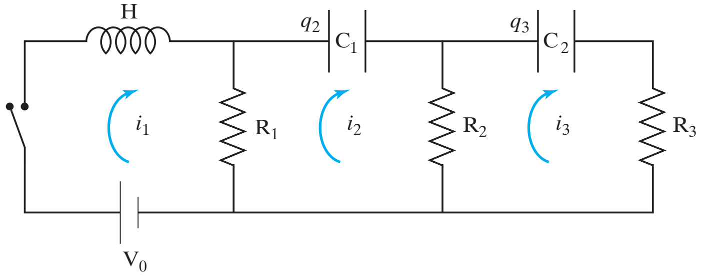


FIGURE 6.12 A three-loop electric circuit with an applied voltage.

an electrical problem leading to a first order system

Applying Kirchhoff's laws (see Section 5.2(d)) to each loop when the switch is closed leads to the three coupled equations

$$H\frac{di_1}{dt} + R_1(i_1 - i_2) = V_0$$
$$R_2 i_2 + R_1(i_2 - i_1) + q_2 C_1 = 0$$
$$R_3 i_3 + R_2(i_3 - i_2) + q_3 C_2 = 0.$$

Using the results $i_2 = dq_2/dt$ and $i_3 = dq_3/dt$ reduces these equations to the coupled system of first order equations

$$H\frac{di_1}{dt} + R_1 i_1 - R_1\frac{dq_2}{dt} = V_0$$
$$(R_1 + R_2)\frac{dq_2}{dt} - R_1 i_1 + q_2 C_1 = 0$$
$$(R_2 + R_3)\frac{dq_3}{dt} - R_2\frac{dq_2}{dt} + q_3 C_2 = 0$$

for i_1, q_2, and q_3. When these are solved the currents i_2 and i_3 follow from $i_2 = dq_2/dt$ and $i_3 = dq_3/dt$.

An example of a different kind is provided by the two degree of freedom vibration system with a damper in Fig. 6.11 that was shown to lead to the two coupled second order equations

$$M\frac{d^2x}{dt^2} = -k(x - y) - Kx - \mu\frac{dx}{dt} + F(t)$$

and

$$m\frac{d^2y}{dt^2} = -k(y - x).$$

Instead of eliminating first y and then x to obtain two fourth order differential equations for x and y, respectively, a different approach is to reduce these two equations to a system of four first order equations by introducing first order derivatives of x and y as new variables.

To do this we set $w = dx/dt$ and $z = dy/dt$, and as a result obtain the simultaneous system of four first order equations

$$\frac{dx}{dt} = w$$
$$\frac{dy}{dt} = z$$
$$M\frac{dw}{dt} + (k + K)x - ky + \mu w = F(t)$$
$$m\frac{dz}{dt} + ky - kx = 0.$$

This reduction of a higher order differential equation, or a coupled system of differential equations, to a first order system is often useful. In Chapter 19 this approach is used when seeking the numerical solution of higher order differential equations by means of the Runge–Kutta method. This method provides accurate numerical solutions of first order differential equations that may be either linear or nonlinear, and it can be adapted to solve higher order differential equations by reducing them to a coupled system of first order equations.

a general homogeneous first order system

A general system of n first order linear variable coefficient differential equations involving the n dependent variables $x_1(t), x_2(t), \ldots, x_n(t)$ that are functions of the independent variable t (in applications t is often the time), the variable coefficients $a_{ij}(t)$, and the nonhomogeneous terms $f_1(t), f_2(t), \ldots, f_n(t)$ has the form

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + f_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + f_2(t) \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ x_n'(t) &= a_{n1}(t)x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}(t)x_n(t) + f_n(t). \end{aligned} \tag{101}$$

System (101) is said to be **homogeneous** when all the functions $f_i(t)$ are zero, and to be **nonhomogeneous** when at least one of them is nonzero. It is a linear system because it is linear in the functions $x_1(t), x_2(t), \ldots, x_n(t)$ and their derivatives, and it is a **variable coefficient** system whenever at least one of the coefficients $a_{ij}(t)$ is a function of t; otherwise, it becomes a **constant coefficient** system.

An **initial value problem** for system (101) involves seeking a solution of (101) such that at $t = t_0$ the variables $x_1(t), x_2(t), \ldots, x_n(t)$ satisfy the initial conditions

$$x_1(t_0) = k_1,\ x_2(t_0) = k_2, \ldots, x_n(t_0) = k_n, \tag{102}$$

where $k_1, k_2, \ldots, k_n$ are given constants.

Matrix notation allows system (101) to be written in the concise form

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t), \tag{103}$$

matrix notation for systems

or more simply as

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b},$$

where a prime again indicates differentiation with respect to t, and the matrices in (103) are defined as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ \vdots \\ x_n'(t) \end{bmatrix},$$

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & \cdots & a_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & \cdots & a_{nn}(t) \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ \vdots \\ f_n(t) \end{bmatrix}. \tag{104}$$

The $n \times 1$ vector $\mathbf{x}(t)$ is called the **solution vector**, the $n \times n$ matrix $\mathbf{A}(t)$ is called the **coefficient matrix**, and the $n \times 1$ vector $\mathbf{b}(t)$ is called the **nonhomogeneous term** of the system.

System (103) becomes an **initial value problem** for the solution $\mathbf{x}(t)$ when at $t = t_0$ the vector $\mathbf{x}(t)$ is required to satisfy the initial condition

$$\mathbf{x}(t_0) = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_n \end{bmatrix}, \tag{105}$$

where $\mathbf{x}(t_0)$ is the **initial vector** and $k_1, k_2, \ldots, k_n$ are given constants.

EXAMPLE 6.25 Express in matrix form the initial value problem

$$\begin{aligned} x_1' &= 2x_1 - x_2 + 4 - t^2 \\ x_2' &= -x_1 + 2x_2 + 1, \quad \text{with } x_1(0) = 1 \quad \text{and} \quad x_2(0) = 0. \end{aligned}$$

Solution The system of equations can be written

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{b}(t)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 4 - t^2 \\ 1 \end{bmatrix},$$

and the initial vector is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As $\mathbf{A}$ is a constant matrix and $\mathbf{b}(t) \neq \mathbf{0}$, this is a constant coefficient nonhomogeneous system. ■

solution by elimination: a first approach

In what follows, our main objective will be to develop matrix methods for the solution of initial value problems for systems of first order linear constant coefficient differential equations. Before developing a matrix approach, we first describe a simple way of solving system (102) when no more than three equations are involved. The method is straightforward and does not use matrix algebra, but it is often useful, and the examples that are solved show that systems can have oscillatory solutions even when no oscillatory term is present in the nonhomogeneous term.

The approach used is called **solution by elimination**, because it involves eliminating all but one of the dependent variables in order to arrive at a single higher order equation for the remaining variable, say $x_1(t)$. Once $x_1(t)$ has been found, it is used in the system of equations to determine sequentially the remaining variables $x_2(t), x_3(t), \ldots, x_n(t)$. The method will be illustrated by means of examples.

EXAMPLE 6.26 Solve by elimination the initial value problem of Example 6.25.

Solution The equations involved are

$$\begin{aligned} x_1' &= 2x_1 - x_2 + 4 - t^2 \\ x_2' &= -x_1 + 2x_2 + 1. \end{aligned}$$

The method will be to eliminate the dependent variable x_2 between the two equations to obtain a single second order equation for x_1. After solving for x_1, the dependent variable x_2 will be found by substituting for x_1 in the first equation. Thus, the solution of this system of two first order equations will involve the solution of a single second order equation, and it will be through this equation that the two arbitrary constants expected to occur in the general solution of the system will enter.

Differentiation of the first equation belonging to the system gives

$$x_1'' = 2x_1' - x_2' - 2t,$$

and after substituting for x_2' from the second equation in the system, this becomes

$$x_1'' = 2x_1' + x_1 - 2x_2 - 1 - 2t.$$

Solving the first equation belonging to the system for x_2 gives

$$x_2 = 2x_1 + 4 - t^2 - x_1',$$

so using this result to eliminate x_2 from the second order equation for x_1 shows that x_1 satisfies the equation

$$x_1'' - 4x_1' + 3x_1 = 2t^2 - 2t - 9.$$

Solving this equation by any method, say by the method of undetermined coefficients, gives

$$x_1(t) = C_1e^{3t} + C_2e^t - \frac{53}{27} + \frac{10}{9}t + \frac{2}{3}t^2,$$

where C_1 and C_2 are arbitrary constants of integration.

It now remains for us to find x_2, and this is accomplished by substituting for x_1 in the first equation, which can be written in the form $x_2 = 2x_1 + 4 - t^2 - x_1'$. As a result we find that

$$x_2(t) = -C_1e^{3t} + C_2e^t - \frac{28}{27} + \frac{8}{9}t + \frac{1}{3}t^2,$$

so the general solution of the nonhomogeneous system is

$$x_1(t) = C_1e^{3t} + C_2e^t - \frac{53}{27} + \frac{10}{9}t + \frac{2}{3}t^2,$$

and

$$x_2(t) = -C_1e^{3t} + C_2e^t - \frac{28}{27} + \frac{8}{9}t + \frac{1}{3}t^2.$$

To solve the initial value problem, C_1 and C_2 must be chosen such that $x_1(0) = 1$ and $x_2(0) = 0$. Setting $t = 0$ in the general solution and using these initial conditions, we find that C_1 and C_2 must satisfy the equations

$$1 = C_1 + C_2 - \frac{53}{27} \quad \text{and} \quad 0 = -C_1 + C_2 - \frac{28}{27},$$

with the solution $C_1 = 26/27$ and $C_2 = 2$. Thus, the required solution of the initial value problem is

$$x_1(t) = \frac{26}{27}e^{3t} + 2e^t - \frac{53}{27} + \frac{10}{9}t + \frac{2}{3}t^2,$$

and

$$x_2(t) = -\frac{26}{27}e^{3t} + 2e^t - \frac{28}{27} + \frac{8}{9}t + \frac{1}{3}t^2.$$

Unlike first order linear differential equations whose complementary function can only contain an exponential function, systems of such equations can give rise to periodic solutions even when these do not occur in the nonhomogeneous term. This is illustrated by the next example.

EXAMPLE 6.27 Solve by elimination the system of differential equations

$$x_1' + 2x_1 - x_2 = 1 + e^{-t}, \quad x_2' + x_1 + 2x_2 = 3,$$

subject to the initial conditions $x_1(0) = 5/2$ and $x_2(0) = -1/2$.

Solution Proceeding as in the previous example by differentiating the first equation with respect to t and substituting for x_2' from the second equation gives

$$x_1'' + 2x_1' + x_1 + 2x_2 - 3 = -e^{-t}.$$

Substituting for x_2 from the first equation belonging to the system then shows that x_1 must satisfy the second order differential equation

$$x_1'' + 4x_1' + 5x_1 = 5 + e^{-t},$$

with the general solution

$$x_1(t) = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t + 1 + (1/2)e^{-t}.$$

Finally, solving the first equation belonging to the system for x_2 and substituting for x_1, we have

$$x_2(t) = -C_1e^{-2t}\sin t + C_2e^{-2t}\cos t + 1 - (1/2)e^{-t}.$$

Thus, the general solution of the system is

$$x_1(t) = C_1e^{-2t}\cos t + C_2e^{-2t}\sin t + 1 + (1/2)e^{-t}$$

and

$$x_2(t) = -C_1e^{-2t}\sin t + C_2e^{-2t}\cos t + 1 - (1/2)e^{-t}.$$

To satisfy the initial conditions, the arbitrary constants C_1 and C_2 must be chosen such that $x_1(0) = 5/2$ and $x_2(0) = -1/2$. Inserting these conditions into the preceding general solution leads to the equations

$$5/2 = C_1 + 3/2 \quad \text{and} \quad -1/2 = C_2 + 1/2, \quad \text{so that } C_1 = 1 \text{ and } C_2 = -1.$$

The solution of the initial value problem is then given by

$$x_1(t) = e^{-2t}(\cos t - \sin t) + 1 + \frac{1}{2}e^{-t}$$

and

$$x_2(t) = -e^{-2t}(\sin t + \cos t) + 1 - \frac{1}{2}e^{-t}.$$

This example illustrates the way in which oscillatory terms can enter into the solution through a higher order equation satisfied by one of the dependent variables, although they may not be present in the nonhomogeneous terms.

As a final example of the elimination method, we consider a homogeneous system of three equations to show how this simple method becomes more difficult when the number of equations is greater than two, and also to demonstrate how care must then be taken with the determination of the arbitrary constants of integration.

EXAMPLE 6.28 Find the general solution of the system of equations

$$x_1' = x_2 + x_3, \quad x_2' = x_1 + x_3, \quad \text{and} \quad x_3' = x_1 + x_2.$$

Solution Differentiating the first equation with respect to t and substituting for x_2' and x_3' from the second and third equations and using the first equation shows that x_1 satisfies the second order equation

$$x_1'' - x_1' - 2x_1 = 0,$$

with the solution

$$x_1(t) = C_1 e^{-t} + C_2 e^{2t},$$

where C_1 and C_2 are arbitrary constants of integration.

Substituting for $x_1(t)$ in the second equation belonging to the system, differentiating the result with respect to t, and then substituting for x_3' from the third equation belonging to the system shows that x_2 satisfies the nonhomogeneous second order equation

$$x_2'' - x_2 = 3C_2 e^{2t},$$

with the solution

$$x_2(t) = C_2 e^{2t} + C_3 e^{-t} + C_4 e^{t}.$$

how to resolve the problem of the arbitrary constants

It now appears that an anomalous situation has arisen, because when seeking a solution of a system of *three* equations, *four* arbitrary integration constants have appeared. This apparent inconsistency will be resolved shortly, so for the moment we continue working with this form of solution for $x_2(t)$.

Subtracting the first two equations belonging to the system gives

$$x_1' - x_2' = x_2 - x_1.$$

After substituting for $x_1(t)$ and $x_2(t)$ in this equation and cancelling terms, this is seen to reduce to $-C_4 e^t = C_4 e^t$. As $e^t \neq 0$ for any t, it follows that $C_4 = 0$, and the apparent inconsistency has been resolved because now only the three arbitrary constants C_1, C_2, and C_3 appear in the general solutions for $x_1(t)$ and $x_2(t)$.

In fact, no further integration is required to determine $x_3(t)$, because substituting $x_1(t)$ and $x_2(t)$ into the first equation belonging to the system and solving for $x_3(t)$ gives

$$x_3(t) = -(C_1 + C_3)e^{-t} + C_2 e^{2t}.$$

Thus, the general solution of the system is given by

$$\begin{aligned} x_1(t) &= C_1 e^{-t} + C_2 e^{2t} \\ x_2(t) &= C_2 e^{2t} + C_3 e^{-t} \\ x_3(t) &= -(C_1 + C_3)e^{-t} + C_2 e^{2t}. \end{aligned}$$

■

Summary

This section has shown how a system of first order equations can arise from a typical electrical problem. A matrix notation for systems was introduced, and an elementary method for solving small systems of equations using elimination was described that avoided the use of matrices. This method was seen to lead to more arbitrary constants in the general solution than the number of equations involved, but a simple argument resolved this difficulty.

EXERCISES 6.9

Solve Exercises 1 through 6 by elimination.

1. $2x_1' = x_1 - x_2,\ 2x_2' = 3x_1 + 5x_2.$

2. $x_1' = -10x_1 - 18x_2,\ x_2' = 6x_1 + 11x_2.$

3. $x_1' = 2x_1 - 12x_2,\ \ 2x_2' = 3x_1 - 8x_2,$ with $x_1(0) = 0$ and $x_2(0) = 1.$

4. $x_1' = 3x_2 + t,\ \ x_2' = 2x_1 + x_2 - 3,$ with $x_1(0) = 1$ and $x_2(0) = 1.$

5. $x_1' = 2x_2 + 4x_3 + 3e^{-t}$, $x_2' = x_1 + x_2 - 2x_3 + 1$, $x_3' = -2x_1 + 5x_3$, with $x_1(0) = 1$, $x_2(0) = 0$, and $x_3(0) = 0$.

6. $x_1' = -2x_1 + 2x_2 + 2x_3 + 3e^t,\ x_2' = -x_1 - x_2 - 2x_3 + 1,$ $x_3' = x_1 + 2x_2 + 3x_3 - 3,$ with $x_1(0) = 1,\ x_2(0) = 1,$ and $x_3(0) = 0.$

6.10 A Matrix Approach to Linear Systems of Differential Equations

We will now consider some general properties of the variable coefficient system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t), \tag{106}$$

where the matrices $\mathbf{x}(t)$, $\mathbf{A}(t)$, and $\mathbf{b}(t)$ are as defined in (103).

A **solution** of system (106) is a vector $\mathbf{x}(t)$ with elements $x_1(t), x_2(t), \ldots, x_n(t)$ that when substituted in system (106) satisfies it identically. Thus, a solution of the initial value problem in Example 6.26 is the vector

a solution in matrix form

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dfrac{26}{27}e^{3t} + 2e^t - \dfrac{53}{27} + \dfrac{10}{9}t + \dfrac{2}{3}t^2 \\ -\dfrac{26}{27}e^{3t} + 2e^t - \dfrac{28}{27} + \dfrac{8}{9}t + \dfrac{1}{3}t^2 \end{bmatrix}.$$

Structure of Solutions of Homogeneous Systems

(a) Linear superposition of solutions

The properties of linear homogeneous systems of differential equations are similar to those of a single linear higher order homogeneous differential equation. A most important property that is common to both is that a linear superposition of solutions of a linear homogeneous system of variable-coefficient first order differential equations is itself a solution of the homogeneous system.

This result is easily proved. Let $\boldsymbol{\Psi}_1(t), \boldsymbol{\Psi}_2(t), \ldots, \boldsymbol{\Psi}_m(t)$ be any m solutions of the linear homogeneous system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t)$, and taking $C_1, C_2, \ldots, C_m$ to be

any set of m arbitrary constants form the vector $\boldsymbol{\Psi}(t) = C_1\boldsymbol{\Psi}_1(t) + C_2\boldsymbol{\Psi}_2(t) + \cdots + C_m\boldsymbol{\Psi}_m(t)$. Then

$$\boldsymbol{\Psi}(t)' = (C_1\boldsymbol{\Psi}_1 + C_2\boldsymbol{\Psi}_2 + \cdots + C_m\boldsymbol{\Psi}_m)' = C_1\boldsymbol{\Psi}_1' + C_2\boldsymbol{\Psi}_2' + \cdots + C_m\boldsymbol{\Psi}_m',$$

so the system $\boldsymbol{\Psi}(t)' = \mathbf{A}(t)\boldsymbol{\Psi}(t)$ becomes

$$\begin{aligned} C_1\boldsymbol{\Psi}_1' + C_2\boldsymbol{\Psi}_2' + \cdots + C_m\boldsymbol{\Psi}_m' &= \mathbf{A}(C_1\boldsymbol{\Psi}_1 + C_2\boldsymbol{\Psi}_2 + \cdots + C_m\boldsymbol{\Psi}_m) \\ &= C_1\mathbf{A}\boldsymbol{\Psi}_1 + C_2\mathbf{A}\boldsymbol{\Psi}_2 + \cdots + C_m\mathbf{A}\boldsymbol{\Psi}_m. \end{aligned}$$

Consequently, as $\boldsymbol{\Psi}_i'(t) = \mathbf{A}(t)\boldsymbol{\Psi}_i(t)$, we have shown that $\boldsymbol{\Psi}(t)$ is also a solution of the homogeneous system, and the result is proved.

(b) Existence and uniqueness

We now state without proof the fundamental theorem on the existence and uniqueness of the solution to the initial value problem for a system of linear variable coefficient first order differential equations. (See, for example, references [3.4] and [3.5].)

THEOREM 6.5

Existence and uniqueness of solutions of linear systems Let the vector $\mathbf{x}(t)$ with the n elements $x_i(t)\,(i = 1, 2, \ldots, n)$ be the solution of the nonhomogeneous variable coefficient system of first order linear differential equations

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t),$$

where the functions $a_{ij}(t)\,(i, j = 1, 2, \ldots, n)$ forming the elements of $\mathbf{A}(t)$ and the elements $f_i(t)\,(i = 1, 2, \ldots, n)$ forming the elements of the vector $\mathbf{b}(t)$ are continuous functions in some interval $a < t < b$. Furthermore, let the elements of $\mathbf{x}(t)$ satisfy the initial conditions $x_i(t_0) = k_i (i = 1, 2, \ldots, n)$, where the k_i are given constants and t_0 is any point such that $a < t_0 < b$. Then the solution of the initial value problem exists and is unique for all t such that $a < t < b$. ■

(c) Fundamental matrix and a test for linear independence of solutions

As with single higher order linear differential equations, the general solution of a homogeneous system will be constructed by the forming a linear combination of all possible linearly independent solutions of the system. For this reason it is necessary to know how many linearly independent solutions belong to a given homogeneous system, and how to test the linear independence of a set of solutions. The answers to these two fundamental questions are provided by the next two theorems, the results of which should be remembered. As the proofs of these theorems may be omitted at a first reading, they are given at the end of this section.

THEOREM 6.6

Linearly independent solutions of a homogeneous system Let the elements $a_{ij}(t)$ $(i, j = 1, 2, \ldots, n)$ of the $n \times n$ matrix $\mathbf{A}(t)$ be continuous in the interval $a < t < b$. Then the linear homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$$

possesses n linearly independent solutions $\mathbf{\Psi}_1(t), \mathbf{\Psi}_2(t), \dots, \mathbf{\Psi}_n(t)$, and every solution of the system is expressible as a linear combination of the form

$$\mathbf{\Psi}(t) = C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t) + \cdots + C_n\mathbf{\Psi}_n(t)$$

for some choice of the constants $C_1, C_2, \dots, C_n$. ■

a fundamental matrix

An $n \times n$ matrix $\mathbf{\Phi}(t)$ whose columns are any n linearly independent solution vectors of the homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ is called a **fundamental matrix** for the system, and Theorem 6.6 shows that the general solution of the system can always be written in the form

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{C},$$

where $\mathbf{C}$ is an n-element column vector with arbitrary constant elements $C_1, C_2, \dots, C_n$.

Clearly, a fundamental matrix is not unique, because any of its columns may be replaced by a linear combination of its columns and the result will remain a fundamental matrix. This follows because if the columns of a determinant are replaced by linear combinations of its columns, the value of the determinant is unaltered, so if initially the determinant was nonsingular, it will remain nonsingular.

THEOREM 6.7

a determinant test for linear independence of solution vectors

Determinant test for the linear independence of solution vectors Let the column vectors $\mathbf{\Psi}_m(t)\,(m=1, 2, \dots, n)$, whose elements $\Psi_1^{(m)}(t), \Psi_2^{(m)}(t), \dots, \Psi_n^{(m)}(t)$, be n solutions of the homogeneous system

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x},$$

in which the elements $a_{ij}(t)\,(i, j = 1, 2, \dots, n)$ of the $n \times n$ matrix $\mathbf{A}(t)$ are continuous functions for $a < t < b$. Then the n vectors $\mathbf{\Psi}_m(t)\,(m = 1, 2, \dots, n)$ are linearly independent solutions for $a < t < b$ if, for some t_0 in the interval, the determinant

$$\Delta(t_0) = \begin{vmatrix} \Psi_1^{(1)}(t_0) & \Psi_1^{(2)}(t_0) & \cdots & \Psi_1^{(n)}(t_0) \\ \Psi_2^{(1)}(t_0) & \Psi_2^{(2)}(t_0) & \cdots & \Psi_2^{(n)}(t_0) \\ \vdots & \vdots & \vdots & \vdots \\ \Psi_n^{(1)}(t_0) & \Psi_n^{(2)}(t_0) & \cdots & \Psi_n^{(n)}(t_0) \end{vmatrix} \neq 0,$$

and the vectors $\mathbf{\Psi}_m(t)\,(m = 1, 2, \dots, n)$ form a basis for solutions of the system. Furthermore, if $\Delta(t_0) \neq 0$, then $\Delta(t) \neq 0$, for all t in $a < t < b$. ■

EXAMPLE 6.29

Find a set of linearly independent solution vectors for the system

$$x_1' = x_2 + x_3, \quad x_2' = x_1 + x_3 \quad \text{and} \quad x_3' = x_1 + x_2,$$

and construct a fundamental matrix.

Solution In Example 6.28 the solution of this system was shown to be

$$x_1(t) = C_1e^{-t} + C_2e^{2t}$$
$$x_2(t) = C_2e^{2t} + C_3e^{-t}$$
$$x_3(t) = -(C_1 + C_3)e^{-t} + C_2e^{2t}.$$

Writing this solution in the form $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{C}$ determined by Theorem 6.7, we obtain

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{2t} & 0 \\ 0 & e^{2t} & e^{-t} \\ -e^{-t} & e^{2t} & -e^{-t} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}.$$

Thus, a fundamental matrix for the system, that is, a matrix whose columns are linearly independent solution vectors of the system, can be taken to be

$$\Phi(t) = \begin{bmatrix} e^{-t} & e^{2t} & 0 \\ 0 & e^{2t} & e^{-t} \\ -e^{-t} & e^{2t} & -e^{-t} \end{bmatrix},$$

provided the solution vectors corresponding to the columns of this matrix are linearly independent. The test for this is provided by Theorem 6.7, and as it is easily shown that $\det \mathbf{\Phi}(t) = -3$. So it follows from Theorem 6.7 that the three column vectors

$$\Psi_1(t) = \begin{bmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{bmatrix}, \quad \Psi_2(t) = \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}, \quad \text{and} \quad \Psi_3(t) = \begin{bmatrix} 0 \\ e^{-t} \\ -e^{-t} \end{bmatrix}$$

are, indeed, linearly independent solution vectors. ■

Proofs of Theorems 6.6 and 6.7

Proof of Theorem 6.6 Consider any set of n linearly independent column vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, each with constant elements, and for some t_0 in $a < t_0 < b$ use them as initial conditions in the set of initial value problems

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{v}_m, \quad \text{for } m = 1, 2, \ldots, n.$$

By the existence and uniqueness theorem, each of these initial value problems has a unique solution $\mathbf{\Psi}_m(t)$ defined on $a < t < b$.

To establish the linear independence of these solutions on $a < t < b$, we suppose, if possible, that constants $C_1, C_2, \ldots, C_n$ can be found such that

$$C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t) + \cdots + C_n\mathbf{\Psi}_n(t) = \mathbf{0}$$

for every t in the interval. Setting $t = t_0$, this result becomes

$$C_1\mathbf{v}_1 + C_2\mathbf{v}_2 + \cdots + C_n\mathbf{v}_n = \mathbf{0},$$

but as the $\mathbf{v}_m$ are linearly independent, this can only be true if $C_1 = C_2 = \cdots = C_n = 0$, so we have proved that the solutions $\mathbf{\Psi}_m(t)$ $(m = 1, 2, \ldots, n)$ are linearly independent over the interval.

We must now show that for some constants $C_1, C_2, \ldots, C_n$, not all of which are zero, every solution of the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$ can be written

$$\mathbf{\Psi}(t) = C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t) + \cdots + C_n\mathbf{\Psi}_n(t),$$

and in particular this result must be true when $t = t_0$.

Define a matrix $\mathbf{\Phi}(t)$ whose columns are the n linearly independent vectors $\mathbf{\Psi}_1(t)$, $\mathbf{\Psi}_2(t), \dots, \mathbf{\Psi}_n(t)$, where the elements of $\mathbf{\Psi}_m(t)$ are $\Psi_1^{(m)}(t), \Psi_2^{(m)}(t), \dots,$ $\Psi_n^{(m)}(t)$, for $m = 1, 2, \dots, n$, so

$$\mathbf{\Phi}(t) = \begin{bmatrix} \Psi_1^{(1)}(t) & \Psi_1^{(2)}(t) & \cdots & \Psi_1^{(n)}(t) \\ \Psi_2^{(1)}(t) & \Psi_2^{(2)}(t) & \cdots & \Psi_2^{(n)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \Psi_n^{(1)}(t) & \Psi_n^{(2)}(t) & \cdots & \Psi_n^{(n)}(t) \end{bmatrix}.$$

Now set $t = t_0$ and consider the matrix equation

$$\mathbf{\Phi}(t_0)\mathbf{C} = \mathbf{\Psi}(t_0),$$

where $\mathbf{C}$ is a column vector with the n elements $C_1, C_2, \dots, C_n$. Expanding the expression on the left and grouping terms shows that

$$\mathbf{\Phi}(t_0)\mathbf{C} = C_1\mathbf{\Psi}_1(t_0) + C_2\mathbf{\Psi}_2(t_0) + \cdots + C_n\mathbf{\Psi}_n(t_0),$$

and so

$$C_1\mathbf{\Psi}_1(t_0) + C_2\mathbf{\Psi}_2(t_0) + \cdots + C_n\mathbf{\Psi}_n(t_0) = \mathbf{\Psi}(t_0).$$

The existence of a unique set of constants $C_1, C_2, \dots, C_n$, not all of which are zero, follows from the fact that $\det\mathbf{\Phi}(t_0) \neq 0$, because of the linear independence of its columns.

As $\mathbf{\Psi}(t)$ and $C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t) + \cdots + C_n\mathbf{\Psi}_n(t)$ are both solutions of the same initial value problem

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{\Psi}(t_0),$$

the existence and uniqueness theorem shows that $\mathbf{\Psi}(t) = C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t)_1 + \cdots + C_n\mathbf{\Psi}_n(t)$ for all t such that $a < t < b$, and the theorem is proved. ■

Proof of Theorem 6.7 The proof is in two parts. First we show that if the vectors are linearly independent, then $\det\mathbf{\Phi}(t) \neq 0$ for all t in the interval. Then we assume the converse, namely that $\mathbf{\Phi}(t)$ is a fundamental matrix, and show this implies $\det\mathbf{\Phi}(t) \neq 0$ for all t in the interval. The fact that every solution of the system can be expressed as a linear combination of the n linearly independent solutions will then follow from Theorem 6.6.

If $\mathbf{\Phi}(t)$ is a matrix whose columns are solution vectors and $\det\,\mathbf{\Phi}(t) \neq 0$, then the vectors are linearly independent. To show this, suppose constants $C_1, C_2, \dots, C_n$ can be found such that

$$C_1\mathbf{\Psi}_1(t) + C_2\mathbf{\Psi}_2(t) + \cdots + C_n\mathbf{\Psi}_n(t) = \mathbf{0}$$

for all t in the interval $a < t < b$. Then for any t_0 in the interval, setting $t = t_0$ the equation can be written

$$\mathbf{\Phi}(t_0)\mathbf{C} = \mathbf{0},$$

where $\mathbf{C}$ is a column matrix with elements $C_1, C_2, \dots, C_n$. As $\det\,\mathbf{\Phi}(t_0) \neq 0$, the only solution of this homogeneous system of algebraic equations is $C_1 = C_2 = \cdots = C_n = 0$, so the column vectors must be linearly independent for all t in the interval.

We must now consider the converse situation and suppose that $\mathbf{\Phi}(t)$ is a fundamental matrix. Then, if $\mathbf{\Psi}(t)$ is a solution of the system, from the definition of

a fundamental solution a unique constant vector $\mathbf{C}$ can always be found such that $\boldsymbol{\Psi}(t) = \boldsymbol{\Phi}(t)\mathbf{C}$ for all t in the interval. To find $\mathbf{C}$ we need only set $t = t_0$ in this last result, because as $\det \boldsymbol{\Phi}(t_0) \neq 0$ the homogeneous system of algebraic equations must have a unique solution. The result is true for each t_0 in the interval, and so it follows that $\det \boldsymbol{\Phi}(t) \neq 0$ over the interval $a < t < b$.

As the set of n vectors $\boldsymbol{\Psi}_m(t)$ $(m = 1, 2, \ldots, n)$ is linearly independent, it follows from Theorem 6.6 that every solution of the system is expressible as a linear combination of these vectors, so they form a basis for solutions of the system. ■

For more information about the material in Sections 6.9 and 6.10 see, for example, references [3.3], [3.4], and [3.16].

Summary

The linear superposition of matrix vector solutions was shown to be permissible, and the concept of a fundamental matrix was introduced, the columns of which contained n linearly independent solution vectors of a linear system of n first order equations. The fundamental matrix had the property that the general solution of the system could be expressed in terms of its product with a column vector containing n arbitrary constants. A determinant test was then developed that established when a set of n solution vectors was suitable to form the columns of a fundamental matrix—that is, to form a basis for the solution set of the system.

EXERCISES 6.10

In Exercises 1 through 6, verify by substitution that the functions $x_1(t)$ and $x_2(t)$ are solutions of the given system of equations. By writing the solution in matrix form, find a fundamental matrix for the system and verify that its columns are linearly independent.

1. $x_1' = x_1 + x_2$, $x_2' = -x_1 + x_2$;
$x_1(t) = e^t(C_1 \cos t + C_2 \sin t)$, $x_2(t) = e^t(C_2 \cos t - C_1 \sin t)$.

2. $x_1' = 2x_1 + x_2$, $x_2' = -2x_1$;
$x_1(t) = e^t(C_1 \cos t + C_2 \sin t)$, $x_2(t) = (C_2 - C_1)e^t \cos t - (C_1 + C_2)e^t \sin t$.

3. $x_1' = x_1 - 2x_2$, $x_2' = x_1 - x_2$;
$x_1(t) = C_1 \cos t + C_2 \sin t$, $x_2(t) = (1/2)(C_1 - C_2)\cos t + (1/2)(C_1 + C_2)\sin t$.

4. $x_1' = -3x_1 - x_2$, $x_2' = 3x_1 + x_2$;
$x_1(t) = C_1 + C_2 e^{-2t}$, $x_2(t) = -3C_1 - C_2 e^{-2t}$.

5. $2x_1' = 2x_1 - x_2$, $x_2' = x_1 + 2x_2$;
$x_1(t) = C_1 e^{3t/2} \cos t/2 + C_2 e^{3t/2} \sin t/2$,
$x_2(t) = -(C_1 + C_2)e^{3t/2} \cos t/2 + (C_1 - C_2)e^{3t/2} \sin t/2$.

6. $2x_1' = -x_1 + x_2$, $x_2' = x_1 - x_2$;
$x_1(t) = C_1 + C_2 e^{-3t/2}$, $x_2(t) = C_1 - 2C_2 e^{-3t/2}$.

6.11 Nonhomogeneous Systems

A nonhomogeneous variable coefficient system of first order linear differential equations can be written

$$\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t). \tag{107}$$

Its general solution can be expressed as the sum of the general solution of the associated homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ that will contain the arbitrary constants, and a *particular solution* free from arbitrary constants that can be taken to be any solution of the nonhomogeneous equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$. This result is recorded and proved in the next theorem.

The Structure of the Solution

THEOREM 6.8

nonhomogeneous system and the structure of the solution

Structure of the solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$ Let $\boldsymbol{\Phi}(t)$ be a fundamental matrix for the homogeneous linear first order system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, and let $\mathbf{P}(t)$ be any solution of the nonhomogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$. Then the **general solution** of the nonhomogeneous system is $\mathbf{x}(t) = \boldsymbol{\Phi}(t)\mathbf{C} + \mathbf{P}(t)$, with $\mathbf{C}$ an n-element column matrix with arbitrary constants $C_1, C_2, \ldots, C_n$ as elements.

Proof The result is almost immediate and follows by substitution. Setting $\mathbf{x} = \boldsymbol{\Phi}(t)\mathbf{C} + \mathbf{P}(t)$, we have $\mathbf{x}' = \boldsymbol{\Phi}'(t)\mathbf{C} + \mathbf{P}'(t)$, so after substitution into the system of differential equations we find that

$$\boldsymbol{\Phi}'(t)\mathbf{C} + \mathbf{P}'(t) = \mathbf{A}\boldsymbol{\Phi}(t)\mathbf{C} + \mathbf{A}\mathbf{P}(t) + \mathbf{b}(t).$$

However, $\boldsymbol{\Phi}'(t)\mathbf{C} = \mathbf{A}(t)\boldsymbol{\Phi}(t)\mathbf{C}$, and by definition $\mathbf{P}(t)$ is any solution of $\mathbf{x}' = \mathbf{A}(t)\mathbf{x} + \mathbf{b}(t)$, so $\mathbf{P}'(t) = \mathbf{A}\mathbf{P}(t) + \mathbf{b}(t)$, showing that substitution of the general solution into the equation leads to an identity, so the theorem is proved. ■

It is important to recognize that solutions of nonhomogeneous linear systems do not have the linear superposition property of solutions of homogeneous systems, and so they do *not* form a vector space.

EXAMPLE 6.30

Find the solution of the initial value problem for the nonhomogeneous system of equations

$$x_1' + 2x_1 + 4x_2 = 1 + 2t, \quad x_2' + x_1 - x_2 = 3t$$

subject to the initial conditions $x_1(0) = 56/9$ and $x_2(0) = -13/9$, and verify the results of Theorem 6.8.

Solution Using the elimination method, the solution of the system can be shown to be

$$x_1(t) = 2/9 + (7/3)t + 2e^{2t} + 4e^{-3t} \quad \text{and} \quad x_2(t) = -4/9 - (2/3)t - 2e^{2t} + e^{-3t},$$

and in matrix form this becomes

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{\mathbf{x}(t)} = \underbrace{\begin{bmatrix} e^{2t} & 4e^{-3t} \\ -e^{2t} & e^{-3t} \end{bmatrix}}_{\boldsymbol{\Phi}(t)} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{C}} + \underbrace{\begin{bmatrix} 2/9 + (7/3)t \\ -4/9 - (2/3)t \end{bmatrix}}_{\mathbf{P}(t)}.$$

Inspection of this form of solution identifies the fundamental matrix $\boldsymbol{\Phi}(t)$ containing exponentials, a column vector $\mathbf{C}$ with elements $C_1 = 2$ and $C_2 = 1$, and a particular solution $\mathbf{P}(t)$ of the nonhomogeneous system represented by the last matrix vector. It is easily checked that the vector $\mathbf{P}(t)$, which contains no constants, is a particular solution of the system. ■

Matrix Methods of Solution

We now describe a number of matrix methods for the solution of both homogeneous and nonhomogeneous constant coefficient systems of linear first order differential equations.

(a) Solution by diagonalization when A has real eigenvalues

solution by diagonalization when eigenvalues are real

Having already illustrated the elementary elimination method for solving a small system of equations, we now describe the powerful and systematic matrix diagonalization method that can be used with systems involving any number of differential equations.

Consider a general nonhomogeneous constant coefficient system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t) \tag{108}$$

where $\mathbf{A}$ is a constant coefficient $n \times n$ matrix with real eigenvalues and n linearly independent eigenvectors. The approach we will use is to try to find a transformation of the dependent variables $x_1, x_2, \ldots, x_n$ forming the elements of vector $\mathbf{x}$, which creates a new set of variables $u_1, u_2, \ldots, u_n$ that form the elements of a vector $\mathbf{u}$ with the property that system (108) can be written as

$$\mathbf{u}' = \mathbf{D}\mathbf{u} + \mathbf{h}, \tag{109}$$

where $\mathbf{D}$ is a diagonal matrix and $\mathbf{h}$ is an n-element column vector with elements that depend on the elements in the nonhomogeneous term $\mathbf{b}(t)$.

If such a transformation can be found, the equations in the system will have been *uncoupled*, because each equation for $u_1, u_2, \ldots, u_n$ can then be solved individually. When $u_1, u_2, \ldots, u_n$ are known, reversing the transformation will give the solution $x_1(t), x_2(t), \ldots, x_n(t)$ of system (108).

Such a transformation has already been provided by Theorem 4.6. It was shown there that if a matrix $\mathbf{P}$ is constructed with the n eigenvectors of $\mathbf{A}$ as its columns, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix with the eigenvalues of $\mathbf{A}$ arranged along its leading diagonal in the same order as the corresponding eigenvectors appear in $\mathbf{P}$.

Adopting this approach, setting

$$\mathbf{x} = \mathbf{P}\mathbf{u}, \tag{110}$$

and substituting in (108) gives

$$\mathbf{P}\mathbf{u}' = \mathbf{A}\mathbf{P}\mathbf{u} + \mathbf{b}(t), \tag{111}$$

where when differentiating $\mathbf{x}(t)$ use has been made of the fact that $\mathbf{P}$ is a constant matrix. The linear independence of the n eigenvectors forming the columns of $\mathbf{P}$ ensures the existence of the inverse matrix $\mathbf{P}^{-1}$, so premultiplying (111) by $\mathbf{P}^{-1}$ gives

$$\mathbf{u}' = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{u} + \mathbf{P}^{-1}\mathbf{b}(t),$$

but $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, so system (108) has been transformed into the uncoupled system

$$\mathbf{u}' = \mathbf{D}\mathbf{u} + \mathbf{P}^{-1}\mathbf{b}(t). \tag{112}$$

The required solution vector $\mathbf{x}(t)$ follows from the result $\mathbf{x}(t) = \mathbf{P}\mathbf{u}$.

Before giving an example, it is necessary to consider whether systems exist for which this method will fail. The answer to this question is not difficult to find,

because the method depends for its success on the diagonalization of $\mathbf{A}$, and this in turn requires that $\mathbf{A}$ have n linearly independent eigenvectors. Consequently, we see that the method will fail if the $n \times n$ matrix $\mathbf{A}$ has fewer than n linearly independent eigenvectors, because then the diagonalizing matrix $\mathbf{P}$ cannot be constructed. This situation occurs when $\mathbf{A}$ has multiple eigenvalues but an eigenvalue with multiplicity r has associated with it fewer than r linearly independent eigenvectors. A typical matrix with this property is

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}.$$

In this case the eigenvalue $\lambda = 1$ occurs with multiplicity 1 and the eigenvalue $\lambda = 0$ with multiplicity 2, but the matrix has only the two linearly independent eigenvectors

$$(\lambda = 1), \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } (\lambda = 0, \text{twice}), \text{ the single eigenvector } \mathbf{x}_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

EXAMPLE 6.31

Use diagonalization to solve the nonhomogeneous system

$$x_1'(t) + 2x_1 + 4x_2 = 2t - 1, \quad x_2'(t) + x_1 - x_2 = \sin t.$$

Solution The system can be written in the form $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}(t)$ with

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -2 & -4 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 2t - 1 \\ \sin t \end{bmatrix}.$$

Matrix $\mathbf{A}$ has the two eigenvalues and eigenvectors

$$\lambda_1 = 2, \ \mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lambda_2 = -3, \quad \mathbf{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

The diagonalizing matrix is thus

$$\mathbf{P} = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}, \quad \text{so} \quad \mathbf{P}^{-1} = \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix},$$

and from the order in which the eigenvectors have been entered as the columns of $\mathbf{P}$, it follows without further computation that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

We have

$$\mathbf{P}^{-1}\mathbf{b}(t) = \begin{bmatrix} 1/5 - (2/5)t + (4/5)\sin t \\ -1/5 + (2/5)t + (1/5)\sin t \end{bmatrix},$$

so, corresponding to (112) the transformed system becomes

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1/5 - (2/5)t + (4/5)\sin t \\ -1/5 + (2/5)t + (1/5)\sin t \end{bmatrix}.$$

In component form these are seen to be the uncoupled equations

$$u_1' = 2u_1 + 1/5 - (2/5)t + (4/5)\sin t$$

and

$$u_2' = -3u_2 - 1/5 + (2/5)t + (1/5)\sin t.$$

The solution of the uncoupled equations is easily shown to be

$$u_1(t) = C_1e^{2t} - (4/25)\cos t - (8/25)\sin t + (1/5)t$$
$$u_2(t) = C_2e^{-3t} - (1/50)\cos t + (3/50)\sin t + (2/15)t - 1/9,$$

where C_1 and C_2 are arbitrary constants. If we use these as the elements of the column vector $\mathbf{U}$, the required solution is given by $\mathbf{x}(t) = \mathbf{Pu}$, and so

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} C_1e^{2t} - (4/25)\cos t - (8/25)\sin t + (1/5)t \\ C_2e^{-3t} - (1/50)\cos t + (3/50)\sin t + (2/15)t - 1/9 \end{bmatrix}.$$

In component form the solution becomes

$$x_1(t) = -4/9 + (1/3)t + (2/25)\cos t + (14/25)\sin t - C_1e^{2t} + 4C_2e^{-3t}$$
$$x_2(t) = -1/9 + (1/3)t - (9/50)\cos t - (13/50)\sin t + C_1e^{2t} + C_2e^{-3t}.$$ ■

(b) Solution by diagonalization when A has complex eigenvalues

solution by diagonalization when eigenvalues are complex

When the diagonalization method is used to solve a system in which $\mathbf{A}$ has pairs of complex conjugate eigenvalues, the approach only differs from the case involving real eigenvalues in that the arbitrary constants introduced at the integration stage are complex. When $\mathbf{A}$ has real coefficients and complex eigenvalues exist, they must do so in complex conjugate pairs, so after integrating an equation corresponding to the complex eigenvalue $\lambda = \alpha + i\beta$, we must introduce a complex integration constant $C_1 + iC_2$. Then, to make the solution real, when integrating the equation corresponding to the complex conjugate eigenvalue $\bar{\lambda} = \alpha - i\beta$ the complex conjugate integration constant $C_1 - iC_2$ must be introduced.

EXAMPLE 6.32

Use diagonalization to solve the system of nonhomogeneous equations

$$x_1'(t) = x_1 + 2x_2 + x_3 + 1$$
$$x_2'(t) = x_2 + x_3 + t$$
$$x_3'(t) = 2x_1 + x_3 + 2t.$$

Solution The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix},$$

and its eigenvalues and eigenvectors are

$$\lambda_1 = 3, \quad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad \lambda_2 = i, \quad \mathbf{x}_2 = \begin{bmatrix} -i \\ 1 \\ -1+i \end{bmatrix}, \quad \lambda_3 = -i, \quad \mathbf{x}_3 = \begin{bmatrix} i \\ 1 \\ -1-i \end{bmatrix}.$$

The diagonalizing matrix

$$\mathbf{P} = \begin{bmatrix} 2 & -i & i \\ 1 & 1 & 1 \\ 2 & -1+i & -1-i \end{bmatrix} \quad \text{and}$$

$$\mathbf{P}^{-1} = \begin{bmatrix} 1/5 & 1/5 & 1/5 \\ -1/10+3i/10 & 2/5-i/5 & -1/10-i/5 \\ -1/10-3i/10 & 2/5+i/5 & -1/10+i/5 \end{bmatrix}.$$

The order in which the columns of **P** are arranged shows without further computation that when diagonalized, **A** will become the matrix

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}.$$

This is because **D** can be written down immediately without the need to calculate $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$, because the order in which the eigenvalues are arranged along the leading diagonal of **D** is the order in which their corresponding eigenvectors form the columns of **A**.

If we write the system as

$$\mathbf{x}' = \mathbf{Ax} + \mathbf{b}(t),$$

with

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad \text{and} \quad \mathbf{b}(t) = \begin{bmatrix} 1 \\ t \\ 2t \end{bmatrix},$$

and set $\mathbf{x}(t) = \mathbf{Pu}$, the system becomes

$$\mathbf{Pu}' = \mathbf{APu} + \mathbf{b}(t),$$

so

$$\mathbf{u}' = \mathbf{P}^{-1}\mathbf{APu} + \mathbf{P}^{-1}\mathbf{b}(t) \quad \text{or} \quad \mathbf{u}' = \mathbf{Du} + \mathbf{P}^{-1}\mathbf{b}(t).$$

A simple calculation then gives

$$\mathbf{P}^{-1}\mathbf{b}(t) = \begin{bmatrix} 1/5+3t/5 \\ -1/10+3i/10+t/5-3it/5 \\ -1/10-3i/10+t/5+3it/5 \end{bmatrix},$$

so writing $\mathbf{u}' = \mathbf{Du} + \mathbf{P}^{-1}\mathbf{b}(t)$ in component form shows that the uncoupled equations become

$$\begin{aligned} u_1'(t) &= 3u_1 + 1/5 + 3t/5 \\ u_2'(t) &= iu_2 - 1/10 + 3i/10 + t/5 - 3it/5 \\ u_3'(t) &= -iu_3 - 1/10 - 3i/10 + t/5 + 3it/5. \end{aligned}$$

Solving the first equation involves no complex numbers and so gives rise to the solution

$$u_1(t) = -2/15 - t/5 + C_1 e^{3t}.$$

However, the other two equations are complex, so remembering that the complex integration constant in the third equation must be the complex conjugate of the one in the second equation leads to the results

$$u_2(t) = 3t/5 - 1/10 - 7i/10 + it/5 + (C_2 + iC_3)(\cos t + i\sin t)$$
$$u_3(t) = 3t/5 - 1/10 + 7i/10 - it/5 + (C_2 - iC_3)(\cos t - i\sin t).$$

Combining these results gives

$$\mathbf{u} = \begin{bmatrix} -2/15 - t/5 + C_1e^{3t} \\ 3t/5 - 1/10 - 7i/10 + it/5 + (C_2 + iC_3)(\cos t + i\sin t) \\ 3t/5 - 1/10 + 7i/10 - it/5 + (C_2 - iC_3)(\cos t - i\sin t) \end{bmatrix},$$

so finally, using $\mathbf{x}(t) = \mathbf{Pu}$, we arrive at the required solution

$$x_1(t) = -5/3 + 2C_1e^{3t} + 2C_2\sin t + 2C_3\cos t$$
$$x_2(t) = -1/3 + t + C_1e^{3t} + 2C_2\cos t - 2C_3\sin t$$
$$x_3(t) = 4/3 - 2t + 2C_1e^{3t} + (2C_3 - 2C_2)\sin t - (2C_2 + 2C_3)\cos t.$$

(c) Solution of a homogeneous system by the matrix exponential

solution using the matrix exponential

For the sake of completeness, we now show how, when $\mathbf{A}$ is diagonalizable, the solution of the homogeneous constant coefficient system $\mathbf{x}' = \mathbf{Ax}$ can be solved by means of the matrix exponential, and we indicate how the method can be extended to enable the solution to be found when $\mathbf{A}$ is *not* diagonalizable. As the Laplace transform method to be described later deals with the solution of initial value problems for linear equations automatically and is simpler to use, the ideas involved will only be outlined. Nevertheless, the matrix exponential is both useful and important when working with systems of equations, so it is necessary to make some mention of it here.

We consider the initial value problem

$$\mathbf{x}' = \mathbf{Ax} \quad \text{subject to the initial condition} \quad \mathbf{x}(t_0) = \mathbf{v}, \tag{113}$$

where $\mathbf{A}$ is an $n \times n$ constant matrix and $\mathbf{v}$ is an arbitrary n-element constant column vector. Then the existence and uniqueness theorem guarantees that a solution certainly exists in some open interval containing t_0. If we define a vector $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{v}$, and set

$$e^{t\mathbf{A}} = \mathbf{I}_n + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \ldots,$$

then

$$\begin{aligned} d\mathbf{x}/dt &= d(e^{t\mathbf{A}})/dt\,\mathbf{v} \\ &= \mathbf{A}\mathbf{e}^{t\mathbf{A}}\mathbf{v} \\ &= \mathbf{Ax}, \end{aligned}$$

so the solution of the initial value problem in (113) can be represented in the form

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{v}. \tag{114}$$

We saw in Section 4.5 that $e^{t\mathbf{A}}$ is easily computed when $\mathbf{A}$ is diagonalizable, but before using this result we first review the ideas that are involved. If $\mathbf{A}$ is diagonalizable to a matrix $\mathbf{D}$, a matrix $\mathbf{P}$ exists such that $\mathbf{A} = \mathbf{PDP}^{-1}$, where the columns of $\mathbf{P}$ are the eigenvectors of $\mathbf{A}$, and the elements of $\mathbf{D}$ are the corresponding eigenvalues of $\mathbf{A}$. Thus, $\mathbf{A}^2 = \mathbf{PDP}^{-1}\,\mathbf{PDP}^{-1} = \mathbf{PD}^2\mathbf{P}^{-1}$, and by extending this argument we have the general result $\mathbf{A}^m = \mathbf{PD}^m\mathbf{P}^{-1}$, for $m = 1, 2 \ldots$. Using this property in the definition of the matrix exponential $e^{t\mathbf{A}}$ given above allows it to be written

$$e^{t\mathbf{A}} = \mathbf{P}[\mathbf{I}_n + t\mathbf{D}\,\frac{t^2}{2!}\mathbf{D}^2 + \frac{t^3}{3!}\mathbf{D}^3 + \ldots]\mathbf{P}^{-1}.$$

Consequently, if

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_n \end{bmatrix},$$

then

$$\mathbf{D}^j = \begin{bmatrix} \lambda_1^j & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2^j & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_n^j \end{bmatrix},$$

and so

$$e^{t\mathbf{A}} = \mathbf{P}\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\lambda_1^j t^j}{j!} & 0 & 0 & \ldots & 0 \\ 0 & \sum_{j=0}^{\infty}\frac{\lambda_2^j t^j}{j!} & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \sum_{j=0}^{\infty}\frac{\lambda_n^j t^j}{j!} \end{bmatrix}\mathbf{P}^{-1},$$

and this shows that

$$e^{t\mathbf{A}} = \mathbf{P}\begin{bmatrix} \exp(\lambda_1 t) & 0 & 0 & \ldots & 0 \\ 0 & \exp(\lambda_2 t) & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \exp(\lambda_n t) \end{bmatrix}\mathbf{P}^{-1}. \tag{115}$$

We have shown that the matrix exponential $e^{t\mathbf{A}}$ is simply another way of representing a fundamental matrix for system (113). So, provided $\mathbf{A}$ can be diagonalized and has real eigenvalues, $e^{t\mathbf{A}}$ can be written down immediately by using result (115).

EXAMPLE 6.33 Use the matrix exponential to solve the system

$$x_1'(t) = -2x_1 + 6x_2$$
$$x_2'(t) = -2x_1 + 5x_2.$$

Solution The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} -2 & 6 \\ -2 & 5 \end{bmatrix},$$

and its eigenvalues and eigenvectors are

$$(\lambda_1 = 1)\,\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad (\lambda_2 = 2)\,\mathbf{x}_2 = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}.$$

The diagonalizing matrix

$$\mathbf{P} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix},$$

and

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

So from (115) we have

$$e^{t\mathbf{A}} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix},$$

and after evaluating the matrix products we obtain

$$e^{t\mathbf{A}} = \begin{bmatrix} 4e^t - 3e^{2t}, & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t}, & -3e^t + 4e^{2t} \end{bmatrix}.$$

Defining a two-element column matrix $\mathbf{C}$ with the arbitrary constants C_1 and C_2 as elements allows the general solution to be written as

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{C} = \begin{bmatrix} 4e^t - 3e^{2t}, & -6e^t + 6e^{2t} \\ 2e^t - 2e^{2t}, & -3e^t + 4e^{2t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

so

$$\mathbf{x}(t) = \begin{bmatrix} (4C_1 - 6C_2)e^t + (6C_2 - 3C_1)e^{2t} \\ (2C_1 - 3C_2)e^t + (4C_2 - 2C_1)e^{2t} \end{bmatrix}.$$

In component form the solution is

$$x_1(t) = (4C_1 - 6C_2)e^t + (6C_2 - 3C_1)e^{2t}$$
$$x_2(t) = (2C_1 - 3C_2)e^t + (4C_2 - 2C_1)e^{2t}.$$

The method applies equally well to the situation in which matrix $\mathbf{A}$ is real but the eigenvalues occur in complex conjugate pairs, as shown by the next example.

EXAMPLE 6.34 Use the matrix exponential to solve the system

$$x_1'(t) = -3x_1 - 4x_2 \quad \text{and} \quad x_2'(t) = 2x_1 + x_2.$$

Solution The matrix $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} -3 & -4 \\ 2 & 1 \end{bmatrix}$$

and its eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are

$$\lambda_1 = -1 + 2\text{i}, \quad \mathbf{x}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 - 2i, \quad \mathbf{x}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}.$$

So

$$\mathbf{P} = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}, \quad \mathbf{P}^{-1} = \begin{bmatrix} -i/2 & 1/2 - i/2 \\ i/2 & 1/2 + i/2 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix},$$

and consequently

$$\begin{aligned} e^{t\mathbf{A}} &= \mathbf{P} \begin{bmatrix} e^{-t}(\cos 2t + i\sin 2t) & 0 \\ 0 & e^{-t}(\cos 2t - i\sin 2t) \end{bmatrix} \mathbf{P}^{-1} \\ &= \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix}. \end{aligned}$$

If we use this expression for $e^{t\mathbf{A}}$ in $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{C}$, the general solution becomes

$$\mathbf{x}(t) = \begin{bmatrix} e^{-t}(\cos 2t - \sin 2t) & -2e^{-t}\sin 2t \\ e^{-t}\sin 2t & e^{-t}(\cos 2t + \sin 2t) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

In component form this reduces to

$$\begin{aligned} x_1(t) &= C_1 e^{-t}\cos 2t - (C_1 + 2C_2)e^{-t}\sin 2t \\ x_2(t) &= (C_1 + C_2)e^{-t}\sin 2t + C_2 e^{-t}\cos 2t. \end{aligned}$$

When $\mathbf{A}$ is not diagonalizable, it is still possible to compute $e^{\mathbf{A}}$ by writing $e^{\mathbf{A}} = e^{\mathbf{K}}e^{\mathbf{L}}$, where $\mathbf{A}$ is the sum of a *diagonal* matrix $\mathbf{K}$ and a *nilpotent* matrix $\mathbf{L}$ (a square matrix that when raised to a finite power becomes the null matrix), because under these circumstances the matrices $e^{\mathbf{K}}$ and $e^{\mathbf{L}}$ commute and $e^{\mathbf{K}+\mathbf{L}} = e^{\mathbf{K}}e^{\mathbf{L}}$. The next example illustrates this approach.

EXAMPLE 6.35 Find $e^{t\mathbf{A}}$ given that

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$$

and use it to solve the homogeneous system

$$x_1'(t) = 4x_1 + x_2 \quad \text{and} \quad x_2'(t) = 4x_2.$$

Solution Matrix $\mathbf{A}$ is not diagonalizable, because the repeated eigenvalue $\lambda = 4$ only gives rise to a single eigenvector. However, $t\mathbf{A}$ can be written as the sum of

the following diagonal matrix $t\mathbf{K}$ and nilpotent matrix $t\mathbf{L}$:

$$t\mathbf{A} = t\mathbf{K} + t\mathbf{L}, \quad \text{where } t\mathbf{K} = \begin{bmatrix} 4t & 0 \\ 0 & 4t \end{bmatrix} \quad \text{and} \quad t\mathbf{L} = \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}.$$

It is easily checked that $(t\mathbf{L})^2 = 0$ and the matrices $t\mathbf{K}$ and $t\mathbf{L}$ commute, so $e^{t\mathbf{A}} = e^{t\mathbf{K}}e^{t\mathbf{L}}$. It follows from this that

$$e^{t\mathbf{K}} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{bmatrix}, \quad \text{and} \quad e^{t\mathbf{L}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

so we arrive at the result

$$e^{t\mathbf{A}} = \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}.$$

The exponential matrix $e^{t\mathbf{A}}$ is a fundamental matrix for the system, so as the general solution is given by $\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{C}$,

$$\mathbf{x}(t) = \begin{bmatrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1e^{4t} + C_2te^{4t} \\ C_2e^{4t} \end{bmatrix}.$$

In component form the solution becomes

$$x_1(t) = C_1e^{4t} + C_2te^{4t} \quad \text{and} \quad x_2(t) = C_2e^{4t}.$$ ■

The nilpotent matrix $\mathbf{L}$ in the last example was seen to give rise to the second linearly independent solution te^{4t} corresponding to the eigenvalue $\lambda = 4$ that occurred with multiplicity 2. If in a larger system with a repeated eigenvalue λ and a nondiagonalizable matrix $\mathbf{A}$ it had been necessary to raise a nilpotent matrix to the power r before it became the null matrix, then in addition to a term of the form $e^{\lambda t}$ appearing in $e^{t\mathbf{A}}$, the repeated eigenvalue would also give rise to the linearly independent terms $te^{\lambda t}, t^2e^{\lambda t}, \ldots, t^{(r-1)}e^{\lambda t}$.

(d) Variation of parameters

A particular integral can be found from the general solution of the homogeneous form of a constant coefficient system by a direct generalization of the method of **variation of parameters** described in Section 6.6. If the system is

the matrix exponential and variation of parameters

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}(t), \tag{116}$$

to find a particular integral $\mathbf{x}_p(t)$ we set

$$\mathbf{x}_p(t) = e^{t\mathbf{A}}\mathbf{u}(t), \tag{117}$$

where the vector $\mathbf{u}(t)$ is to be determined.

Then, as $\mathbf{x}_p'(t) = \mathbf{A}e^{t\mathbf{A}}\mathbf{u}(t) + e^{t\mathbf{A}}\mathbf{u}'(t)$, substituting for $\mathbf{x}_p(t)$ in system (106) gives

$$\mathbf{A}e^{t\mathbf{A}}\mathbf{u}(t) + e^{t\mathbf{A}}\mathbf{u}'(t) = \mathbf{A}e^{t\mathbf{A}}\mathbf{u}(t) + \mathbf{b}(t),$$

so after cancelling the terms $\mathbf{A}e^{t\mathbf{A}}\mathbf{u}(t)$ and premultiplying the result by $e^{-t\mathbf{A}}$, the inverse of $e^{t\mathbf{A}}$ because $e^{t\mathbf{A}}$ and $e^{-t\mathbf{A}}$ commute, we find that

$$\mathbf{u}'(t) = e^{-t\mathbf{A}}\mathbf{b}(t), \tag{118}$$

from which $\mathbf{u}(t)$ now follows.

In the equation for $\mathbf{u}(t)$ the matrix exponential $e^{-t\mathbf{A}}$ is determined from $e^{t\mathbf{A}}$ by changing the sign of t. The expression on the right of (118) is simply a column vector with elements that are known functions of t, so the components of (118) can be integrated separately to find the elements $u_1(t), u_2(t), \ldots, u_n(t)$ of $\mathbf{U}(t)$. Then, when $\mathbf{U}(t)$ is known, the particular integral follows from (117).

The general solution of (116) is the sum of the solution of the homogeneous form of the system and the particular integral $\mathbf{x}_p(t)$.

EXAMPLE 6.36 Use the method of variation of parameters to solve the nonhomogeneous system.

$$x_1'(t) = -2x_1 + 6x_2 + t$$
$$x_2'(t) = -2x_1 + 5x_2 - 1.$$

Solution The homogeneous form of this system was obtained in Example 6.33, where it was shown that

$$e^{t\mathbf{A}} = \begin{bmatrix} 4e^{t} - 3e^{2t} & -6e^{t} + 6e^{2t} \\ 2e^{t} - 2e^{2t} & -3e^{t} + 4e^{2t} \end{bmatrix},$$

so

$$e^{-t\mathbf{A}} = \begin{bmatrix} 4e^{-t} - 3e^{-2t} & -6e^{-t} + 6e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -3e^{-t} + 4e^{-2t} \end{bmatrix}.$$

As

$$\mathbf{b}(t) = \begin{bmatrix} t \\ -1 \end{bmatrix}, \text{ we have } e^{-t\mathbf{A}}\mathbf{b}(t) = \begin{bmatrix} 2(3+2t)e^{-t} - 3(2+t)e^{-2t} \\ (3+2t)e^{-t} - 2(2+t)e^{-2t} \end{bmatrix},$$

but $\mathbf{u}'(t) = e^{-t\mathbf{A}}\mathbf{b}(t)$, so

$$u_1'(t) = 2(3+2t)e^{-t} - 3(2+t)e^{-2t}$$
$$u_2'(t) = (3+2t)e^{-t} - 2(2+t)e^{-2t}.$$

When these equations are integrated, the arbitrary constants of integration can be set equal to zero, because if they are nonzero the terms they introduce are of the same type as the solution of the homogeneous system, so they can be absorbed into it. As a result, integration gives

$$u_1(t) = -2(5+2t) + \frac{3}{2}\left(\frac{5}{2} + t\right)e^{-2t}$$

and

$$u_2(t) = -(5+2t)e^{-t} + \left(\frac{5}{2} + t\right)e^{-2t}.$$

Finally, if we set $\mathbf{x}_p(t) = e^{t\mathbf{A}}\mathbf{u}(t)$ with $u_1(t)$ and $u_2(t)$ taken as the elements of $\mathbf{u}(t)$, the particular integral becomes

$$\mathbf{x}_p(t) = \begin{bmatrix} -\dfrac{25}{4} - \dfrac{5}{2}t \\ -\dfrac{5}{2} - t \end{bmatrix}.$$

The solution $\mathbf{x}_c(t)$ of the homogeneous system (the complementary function) found in Example 6.33 was

$$\mathbf{x}_c(t) = \begin{bmatrix} (4C_1 - 6C_2)e^t + (6C_2 - 3C_1)e^{2t} \\ (2C_1 - 3C_2)e^t + (4C_2 - 2C_1)e^{2t} \end{bmatrix},$$

so the solution of the nonhomogeneous system $\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t)$ is given by

$$x_1(t) = (4C_1 - 6C_2)e^t + (6C_2 - 3C_1)\mathrm{e}^{2t} - \frac{25}{4} - \frac{5}{2}t$$

$$x_2(t) = (2C_1 - 3C_2)e^t + (4C_2 - 2C_1)e^{2t} - \frac{5}{2} - t.$$

■

General Remark

The way in which combinations of arbitrary constants appear when we multiply functions in the general solution of a homogeneous system of differential equations is determined by the method of solution. So, for example, when we solve a system by elimination, the choice of variable to be eliminated first will influence the form of the result, as will the ordering of the eigenvectors when diagonalizing the matrix **A**. A combination of arbitrary constants is simply an arbitrary constant, though the ratio of all similar combinations of constants multiplying corresponding functions in different forms of the solution must be the same.

This can be illustrated by considering the solution of the homogeneous form of the equation in Example 6.36 that was found to be

$$x_1(t) = (4C_1 - 6C_2)e^t + (6C_2 - 3C_1)e^{2t}$$

$$x_2(t) = (2C_1 - 3C_2)e^t + (4C_2 - 2C_1)e^{2t}.$$

This solution can be written in an equivalent but different-looking form by setting $K_1 = 2C_1 - 3C_2$ and $K_2 = 6C_2 - 3C_1$, where K_1 and K_2 are themselves arbitrary constants. After changing the constants in this manner the solution becomes

$$x_1(t) = 2K_1e^t + K_2e^{2t}$$

and

$$x_2(t) = K_1e^t + \frac{2}{3}K_2e^{2t},$$

and other equivalent forms are also possible.

The above remarks should be remembered when comparing solutions to problem sets with the solutions given at the end of the book. As a particular integral contains no arbitrary constants, its form remains the same irrespective of the manner in which it has been determined.

An account of the material in this section is to be found in references [3.5] and [3.15].

Summary

The structure of the solution of a linear nonhomogeneous system of equations was explained, and a matrix method of solution was developed for constant coefficient systems that depended on the diagonalization of the coefficient matrix. The cases of real and complex eigenvalues of the coefficient matrix were examined separately, and it was shown how systems of equations with real coefficient matrices can lead to solutions involving trigonometric functions. A different method of solution was then developed using the concept of the matrix exponential.

EXERCISES 6.11

In Exercises 1 through 6 find a fundamental matrix and the general solution of the system.

1. $x_1' = -x_2, x_2' = 2x_1$.

2. $x_1' = -x_1 - 5x_2, x_2' = x_1 - 5x_2$.

3. $x_1' = -3x_1 - 4x_2, x_2' = 2x_1 + x_2$.

4. $x_1' = -x_1 - 4x_2, x_2' = x_1 + 4x_2$.

5. $x_1' = 2x_2, x_2' = -2x_3, x_3' = 2x_2$.

6. $x_1' = -3x_2, x_2' = -3x_3, x_3' = 3x_2$.

In Exercises 7 through 18 find the general solution of the system by diagonalization.

7. $x_1' = -10x_1 - 18x_2 + t, x_2' = 6x_1 + 11x_2 + 3$.

8. $x_1' = -2x_2 + \sin t, x_2' = -2x_1 - t$.

9. $x_1' = x_1 - x_2 + \cos t, x_2' = -x_1 + x_2 + e^{3t}$.

10. $x_1' = x_2 + e^{-t}, x_2' = -x_1 + 2x_2 - 4$.

11. $x_1' = 2x_1 + 3x_2 - \sin t, x_2' = x_1 - 2x_2$.

12. $x_1' = -x_1 - 2x_2 + \cos t, x_2' = x_1 + x_2 + 4$.

13. $x_1' = -2x_1 + 2x_2 + 2x_3 + \sin t, x_2' = -x_2 + 3, x_3' = -2x_1 + 4x_2 + 3x_3$.

14. $x_1' = x_1 + 2x_2 + 3 + 2t, x_2' = x_2 + t, x_3' = 2x_1 + x_3 + 1$.

15. $x_1' = x_1 + 2x_2 + x_3 + t, \quad x_2' = x_2 - x_3 + 2, \quad x_3' = 2x_1 + x_3 + 2t$.

16. $x_1' = x_2 + t, x_2' = x_3, x_3' = x_2$.

17. $x_1' = x_1 + 2x_2 + x_3 + 2e^{-t}, x_2' = x_2 + x_3 + t, x_3' = 2x_1 + x_3 + 2t$.

18. $x_1' = x_2 + 5, x_2' = x_3 + t, x_3' = x_2 + 2t$.

Solve Exercises 19 through 26 by means of the matrix exponential.

19. $2x_1' = x_1 - x_2, 2x_2' = 3x_1 + 5x_2$.

20. $x_1' = -10x_1 - 18x_2, x_2' = 6x_1 + 11x_2$.

21. $x_1' = -x_2, x_2' = 2x_1$.

22. $x_1' = 2x_1 - 12x_2, 2x_2' = 3x_1 - 8x_2$.

23. $x_1' = 7x_1 - 34x_2, x_2' = 2x_1 - 9x_2$.

24. $x_1' = -x_1 - 5x_2, x_2' = x_1 - 5x_2$.

25. $x_1' = -3x_1 - 4x_2, x_2' = 2x_1 + x_2$.

26. $x_1' = -x_1 + 2x_2, x_2' = x_1 + x_2$.

Solve Exercises 27 through 30 by the method of variation of parameters.

27. $x_1' = 10x_1 + 18x_2 + \sin t, x_2' = -6x_1 - 11x_2 + t$.

28. $x_1' = -x_2 + 3e^{4t}, x_2' = -2x_1 + x_2 - 2$.

29. $x_1' = 3x_1 + 4x_2, x_2' = -2x_1 - x_2 - t^2$.

30. $x_1' = -x_2 + 5, x_2' = x_1 + 2x_2 - 1$.

Solve the initial value problems 31 through 36 by any of the methods in this chapter.

31. $x_1' = x_2 + 1, x_2' = 2x_1 - x_2 + t$, with $x_1(0) = 1, x_2(0) = 0$.

32. $x_1' = 3x_2 + t, x_2' = 2x_1 + x_2 - 3$, with $x_1(0) = 1, x_2(0) = 1$.

33. $x_1' = 2x_1 + x_2 - e^t, x_2' = -2x_1 - x_2 - 3$, with $x_1(0) = 0, x_2(0) = 1$.

34. $x_1' = -3x_1 - x_2 + 3t, x_2' = x_1 - x_2 - 3$, with $x_1(0) = 1, x_2(0) = 3$.

35. $x_1' = -3x_1 - 5x_2 - 12x_3 + \sin t, \quad x_2' = -2x_1 + 1, \quad x_3' = x_1 + x_2 + 2x_3 - t$, with $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1$.

36. $x_1' = -2x_1 + 2x_2 + 2x_3 + 3e^t, \quad x_2' = -x_1 - x_2 - 2x_3 + 1, x_3' = x_1 + 2x_2 + 3x_3 - 3$, with $x_1(0) = 1, \quad x_2(0) = 1, x_3(0) = 0$.

6.12 Autonomous Systems of Equations

Autonomous Systems, the Phase Plane, Stability, and Linear Systems

The general form of a nonlinear system of two simultaneous first order differential equations for the functions $x(t)$, $y(t)$ that depend on the time t is

$$\begin{aligned} \frac{dx}{dt} &= f_1(x, y, t) \\ \frac{dy}{dt} &= g_1(x, y, t). \end{aligned} \tag{119}$$

This system is linear and nonhomogeneous if

$$f_1(x, y, t) = a(t)x(t) + b(t)y(t) + h(t)$$

and

$$g_1(x, y, t) = c(t)x(t) + d(t)y(t) + k(t),$$

and homogeneous if, in addition, $h(t) = k(t) \equiv 0$.

If the dependence of the functions f_1 and g_1 on the time t is only through the functions $x(t)$ and $y(t)$, the time dependence is implicit and $f_1 = f(x, y)$ and $g_1 = g(x, y)$, causing the system of equations in (119) to become

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y). \tag{120}$$

autonomous and nonautonomous systems

Systems of this type are called **autonomous**, and they describe physical phenomena such as chemical reactions that, provided all conditions remain the same, will yield identical results whenever the reactions are repeated. It is because of this that autonomous systems are sometimes said to be **time invariant** systems. This situation should be contrasted with the **nonautonomous** behavior of an electrical circuit containing temperature-dependent elements that will cause its behavior to vary as the ambient temperature changes with time.

equilibrium or critical point

A point (x_0, y_0) where both of the derivatives dx/dt and dy/dt in (120) vanish, so that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 0,$$

is called an **equilibrium point** or a **critical point** of the system.

trajectories or paths

If the differential equations in (120) are solved subject to the initial conditions $x_0 = x(t_0)$, $y_0 = y(t_0)$ imposed at time $t = t_0$, it is convenient to regard $(x(t), y(t))$ as a point in the (x, y)-plane that traces out a curve as t increases. Such curves, along which the time t can be regarded as a parameter, are called **trajectories** or **paths**, and sometimes **orbits** in the (x, y)-plane. The (x, y)-plane itself is then called the **phase plane**. Associated with each trajectory is the direction in which the point $(x(t), y(t))$ moves as t increases, and in the phase plane these directions are usually indicated by adding arrows to trajectories. The pattern of trajectories associated with a given autonomous system of equations is called the **phase portrait** of the system.

phase portrait

The reason why in autonomous systems the time t can be regarded as a parameter can be seen by dividing the second equation in (120) by the first to obtain the differential equation

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}, \tag{121}$$

in which t is absent. Had the **nonautonomous** system of equations in (119) been treated in similar fashion, dy/dx would have exhibited an explicit dependence on the time.

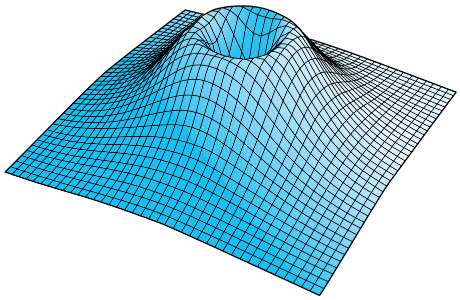


FIGURE 6.13 A depression in a surface surrounded by an elevated rim.

At an equilibrium point (x_0, y_0) of the system in (120), the vanishing of both f and g causes dy/dx in (121) to become *indeterminate* at that point, so initial conditions imposed at an equilibrium point cannot determine a unique solution. This has the effect that on passing through an equilibrium point, a point moving along one trajectory can move onto a different trajectory.

At an equilibrium point of an autonomous system, a physical system represented by the equations is in an equilibrium state. This state is said to be **stable** if, when the system is subjected to arbitrarily small disturbances, it always remains in the neighborhood of the same equilibrium state. If, however, the result of arbitrarily small disturbances is to make the system change to a different equilibrium state, to make the displacement grow unrestrictedly, or, depending on the displacement, to make the system sometimes return to the original equilibrium state and sometimes to cause the displacement increase unrestrictedly, the state is said to be **unstable**.

stability, instability, and asymptotic stability

A dynamical analogy illustrating stable and unstable situations is provided by considering Fig. 6.13, which represents a depression in a surface surrounded by an elevated rim, beyond which the level of the surface falls away steadily. A ball placed at the bottom of the depression is in a stable equilibrium state, because after any small displacement gravity will cause it to try to return to the equilibrium state. If, however, the displacement is large the motion will be unstable, because the ball will leave the depression and roll away indefinitely as time increases. Every point on the top of the rim represents an unstable equilibrium state because, depending on the direction of the displacement, the ball may move to another point on the rim, return to the depression, or roll away indefinitely. So this system has one stable equilibrium state at the bottom of the depression, and an infinite number of unstable states around the top of the rim.

Stability and asymptotic stability

The notion of stability can be made more precise by introducing the function $\Delta(t)$ that measures the distance in the phase plane of a point $(x(t), y(t))$ on a trajectory at time t from an equilibrium point at (x_0, y_0), where

$$\Delta(t) = \sqrt{(x(t) - x_0)^2 + (y(t) - y_0)^2}.$$

(i) The equilibrium point (x_0, y_0) is said to be **stable** if for every arbitrarily small number $\varepsilon > 0$, a number $\delta > 0$ can be found such that if $\sqrt{x_0^2 + y_0^2} < \delta$, then $\Delta(t) < \varepsilon$ for all time t.

(ii) The equilibrium point (x_0, y_0) is said to be **asymptotically stable** if it is stable in the sense of (i) and a number α can be found such that if $\sqrt{x_0^2 + y_0^2} < \alpha$, then $\Delta(t) \to 0$ as $t \to \infty$.

The implication of these definitions is that when an equilibrium point (x_0, y_0) is stable, a trajectory starting close to (x_0, y_0) will remain close to it, but if the point is asymptotically stable any trajectory starting close to (x_0, y_0) will eventually converge to the equilibrium point as $t \to \infty$. Asymptotically stable equilibrium points can be said to *attract* trajectories, so such points are called **attractors**, whereas equilibrium points from which the distance function $\Delta(t)$ increases without bound as t increases are said to **repel** trajectories.

In the dynamical example just given, in the absence of friction, the point at the bottom of the depression will be a *stable* state, because after a small displacement the ball will forever move around the lowest point. If, however, friction is present, the lowest point of the depression will be an *asymptotically stable* state, because after any small displacement the ball will eventually come to rest at the lowest point.

Interest in autonomous systems centers around the fact that trajectories in phase space provide qualitative information about the entire class of solutions of the system and, in particular, about properties of solutions when f and g are nonlinear and no analytical solution can be found.

predator–prey problem

A classical example of a nonlinear autonomous system is the **predator–prey** system of equations introduced and studied by Volterra and Lotka around 1930. They considered the ecological situation in which an isolated colony of foxes and rabbits coexist, with the foxes eating the rabbits and the rabbits feeding on a plentiful supply of vegetation. When the rabbits are numerous, the foxes are well fed and their numbers will grow, but when the number of foxes increases to the point where the rabbit population declines, the number of foxes will begin to fall, giving the rabbit population an opportunity to regenerate. This process, it was postulated, could explain the nonlinear cyclic variation in fox and rabbit populations that is observed in nature. This predator–prey model involving foxes and rabbits will become *nonautonomous* if some external factors are introduced that reduce the fox and rabbit populations by some other means.

To derive the predator–prey equations, let $x(t)$ be the number of rabbits present at time t. Then, as vegetation is plentiful, without foxes the rabbit population will grow at a rate proportional to the number of rabbits, so we can write

$$\frac{dx}{dt} = ax,$$

where $a > 0$ is a constant. Assuming that the rate at which foxes eat rabbits is proportional to the product of the number of rabbits $x(t)$ and the number of foxes $y(t)$ present at time t, the rabbit population described by the preceding equation must be modified to allow for this reduction, and so it becomes

$$\frac{dx}{dt} = ax - bxy,$$

where $b > 0$ is a constant.

The differential equation governing the fox population $y(t)$ is derived in a similar manner, but now the number of foxes *decreases* as the rabbit population decreases, leading to a differential equation of the form

$$\frac{dy}{dt} = -cy + dxy,$$

where $c > 0$ and $d > 0$ are constants. The classical **predator–prey** equations are the two nonlinear autonomous equations

$$\begin{aligned} \frac{dx}{dt} &= x(a - by) \\ \frac{dy}{dt} &= y(xd - c). \end{aligned} \tag{122}$$

This nonlinear autonomous system has no analytical solution, so either individual solutions must be found by numerical computation (see Section 19.7), or phase-plane methods must be used to determine the qualitative behavior of solutions of this system. An obvious feature of the predator–prey system of equations is that an equilibrium state exists when $dx/dt = dy/dt = 0$, and this occurs at the origin $(0, 0)$ and when $x = c/d$ and $y = a/b$. The first equilibrium state is of no interest because then neither rabbits nor foxes are present, but in the other equilibrium state the rabbit and fox populations will remain static, though deviations from this situation can be expected to initiate nonlinear oscillations in the population numbers.

The predator–prey model, although simple and developed initially for ecological reasons, can be modified and applied to other situations such as the spread of an infectious disease, the competition between industries for a raw material that is in limited supply, or when industries compete for the same market.

linearization

When the functions $f(x, y)$ and $g(x, y)$ in (120) are nonlinear, or are complicated in other ways, to help understand the behavior of the system the functions f and g are often **linearized** about an equilibrium point at (x_0, y_0) that is of interest. This involves expanding f and g about (x_0, y_0) as two-variable Taylor series expansions, and then replacing f and g in (120) by the linear terms in these expansions.

If, for example, (x_0, y_0) is an equilibrium point of the system of equations in (120), then $f(x_0, g_0) = 0$ and $g(x_0, y_0) = 0$, and expanding f and g about the point (x_0, y_0) gives

$$f(x, y) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \text{higher order terms}$$

and

$$g(x, y) = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0) + \text{higher order terms.}$$

Substituting only the first order terms from these expansions into system (120) simplifies it to the constant coefficient linear autonomous system

$$\frac{d(x - x_0)}{dt} = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and

$$\frac{d(y - y_0)}{dt} = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0). \tag{123}$$

Setting $X = x - x_0$ and $Y = y - x_0$, we can write these equations in the matrix form

$$\frac{d\mathbf{z}}{dt} = \mathbf{J}(x_0, y_0)\mathbf{z}, \tag{124}$$

where

$$\mathbf{z} = \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \mathbf{J}(x_0, y_0) = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}.$$

Jacobi matrix of the system

The matrix $\mathbf{J}(x_0, y_0)$ is called the **Jacobi matrix** of the system at the point (x_0, y_0), and we will see later how the eigenvalues of $\mathbf{J}(x_0, y_0)$ determine the nature of the equilibrium point at (x_0, y_0).

It is reasonable to suppose that when the neglected remainder terms in the Taylor series expansions of f and g are suitably small, the behavior of this linearized system of equations in some neighborhood of the equilibrium point at (x_0, y_0) will be qualitatively similar to that of the original nonlinear system.

As an illustration of the linearization process, let us now linearize the predator–prey equations in (122) about the equilibrium point at $x = c/d$ and $y = a/b$. Identifying $f(x, y)$ with $x(a - by)$ and $g(x, y)$ with $y(dx - c)$, substituting into the Jacobian $\mathbf{J}(x_0, y_0)$ with $x_0 = c/d$ and $y_0 = a/b$, and setting $X = x - c/d$ and $Y = y - a/b$ leads to the linearized predator–prey equations

$$\begin{aligned} \frac{dX}{dt} &= -\frac{bc}{d}Y \\ \frac{dY}{dt} &= \frac{ad}{b}X. \end{aligned} \tag{125}$$

These equations are easily integrated to give the following equation for the trajectories in the (X, Y) phase plane:

$$X^2 + (cb^2/ad^2)Y^2 = k^2, \quad \text{where } k \text{ is an integration constant.}$$

Reverting to the original variables shows that after linearization, each trajectory in the (x, y) phase plane that is close to the equilibrium point is a member of the family of ellipses

$$(x - c/d)^2 + (cb^2/ad^2)(y - a/b)^2 = k^2, \tag{126}$$

which have their common center at the point $(c/d, a/b)$ in the (x, y) phase plane. This shows that in a neighborhood of the equilibrium point the *phase portrait* of the predator–prey system can be expected to be approximated by this family of ellipses.

This result indicates that close to the equilibrium condition, the rabbit and fox populations can be expected to exhibit a cyclic variation with respect to time. This conclusion follows from the fact that as the time t increases, starting at an initial point on a trajectory where $x_0 = x(t_0)$, $y_0 = y(t_0)$ at a time $t = t_0$, the point $(x(t), y(t))$ will move around the ellipse that passes through this point until after a suitable interval of time it returns to its starting point. In this case linearization has produced elliptical trajectories centered on the equilibrium point, so in the nonlinear case the trajectories can be expected to be distorted ellipses.

Before considering nonlinear autonomous systems we will determine the nature of the equilibrium points associated with the general linear two variable

autonomous system, which in standard notation can be written

$$\begin{aligned}\frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + yd,\end{aligned} \tag{127}$$

where a, b, c, and d are constants, and the second term dy on the right of the second equation is *not* to be confused with the differential dy.

Setting $dx/dt = dy/dt = 0$ in (127) and solving for x and y shows the origin to be the *only* equilibrium point if

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0. \tag{128}$$

When the (127) is integrated once, it yields what is called a **first integral** of the system. A first integral is *not* a solution of the system, because although it is an equation that connects $x(t)$ and $y(t)$, it does not express either function explicitly in terms of t. First integrals are useful because they are easier to obtain than solutions of general autonomous systems, and they provide qualitative information about the general behavior of the set of all solutions. This can be seen from the first integral of the linearized predator–prey system in (125) because, although this did not yield a solution in terms of t, it did confirm that the linearized system exhibits a periodic behavior of the two populations in a neighborhood of the equilibrium point.

A simple example of a linear autonomous system can be derived from any physical system, be it electrical, mechanical, or otherwise, that can be represented by the homogeneous constant-coefficient second order equation

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0. \tag{129}$$

Setting $dy/dt = x$, we can write the second order equation as the linear autonomous system

$$\begin{aligned}\frac{dx}{dt} &= -ax - by \\ \frac{dy}{dt} &= x,\end{aligned} \tag{130}$$

with t as a parameter, or to the equivalent variables separable equation

$$\frac{dy}{dx} = -\frac{x}{ax + by}, \tag{131}$$

where now only x and y are present.

As a special case, when $a = 0$ and $b = n^2$, result (131) becomes

$$\frac{dy}{dx} = -\frac{x}{n^2 y},$$

for which a first integral is seen to be

$$x^2 + n^2y^2 = k^2.$$

This represents a family of elliptical trajectories all centered on the equilibrium point of the system that is located at the origin. The argument used earlier in connection with the linearized predator–prey equations shows that solutions of the system (131) when $a = 0$ and $b = n^2$ must be periodic. This is to be expected, because

with these values of a and b equation (129) describes undamped simple harmonic oscillations. In this simple case, as $x = dy/dt$,

$$\frac{dy}{\sqrt{k^2 - n^2 y^2}} = dt,$$

and after integration this gives

$$y(t) = (k/n)\sin[n(t + t_0)],$$

which is the general solution of (129) for $a = 0$ and $b = x^2$.

When we considered the linearized predator–prey equations, the family of ellipses around the equilibrium point that were found represented an *approximation* to the phase portrait of the system in a *neighborhood* of the equilibrium point. In this case, however, system (130) is linear, so no linearization is involved and the family of elliptical trajectories forms the true phase portrait of system (130).

The linear autonomous system (127) can be written in the matrix form

$$d\mathbf{x}/dt = \mathbf{J}\mathbf{x}, \tag{132}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{133}$$

This system was studied in detail in Section 6.10, where it was seen that its solution depends on the eigenvalues of $\mathbf{J}$ determined by the characteristic equation

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc) = 0. \tag{134}$$

Setting $\alpha = a + d$ and $\beta = ad - bc$, the characteristic equation in (134) becomes

$$\lambda^2 - \alpha\lambda + \beta = 0, \tag{135}$$

with the discriminant $\Delta = (a - d)^2 + 4bc$.

The pattern of the trajectories of the autonomous system in (132), equivalently (127), is determined completely by the eigenvalues λ_1 and λ_2 of $\mathbf{J}$ and their associated eigenvectors: that is to say, by the fundamental solutions of the system. If the eigenvalues are real and $\lambda_1 \neq \lambda_2$, a matrix $\mathbf{P}$ can always be found that simplifies the system by reducing $\mathbf{J}$ to a diagonal matrix $\mathbf{D}$ through the result $\mathbf{P}^{-1}\mathbf{J}\mathbf{P} = \mathbf{D}$, with λ_1 and λ_2 the elements on the leading diagonal of $\mathbf{D}$ (see Section 4.2). The transformation $\mathbf{x} = \mathbf{P}\mathbf{u}$ with $\mathbf{u} = [u, v]^{\mathrm{T}}$ then reduces (132) to the simpler form $d\mathbf{u}/dt = \mathbf{D}\mathbf{u}$, showing that $du/dt = \lambda_1 u$ and $dv/dt = \lambda_2 v$.

These equations have the general solution

$$u = Ae^{\lambda_1 t} \quad \text{and} \quad v = Be^{\lambda_2 t}, \tag{136}$$

so the form of the trajectories about the equilibrium point at the origin in the (u, v) phase plane is seen to depend on both the signs of the eigenvalues λ_1 and λ_2 and their magnitudes.

When the discriminant $\Delta > 0$, the eigenvalues λ_1 and λ_2 will be real, and then there are three cases to consider.

(i) Unstable nodes: λ_1 and λ_2 are positive

Examination of the solution in (136) shows that the trajectories must take one of the two forms illustrated in Figs. 6.14a and 6.14b. In this case the equilibrium point at

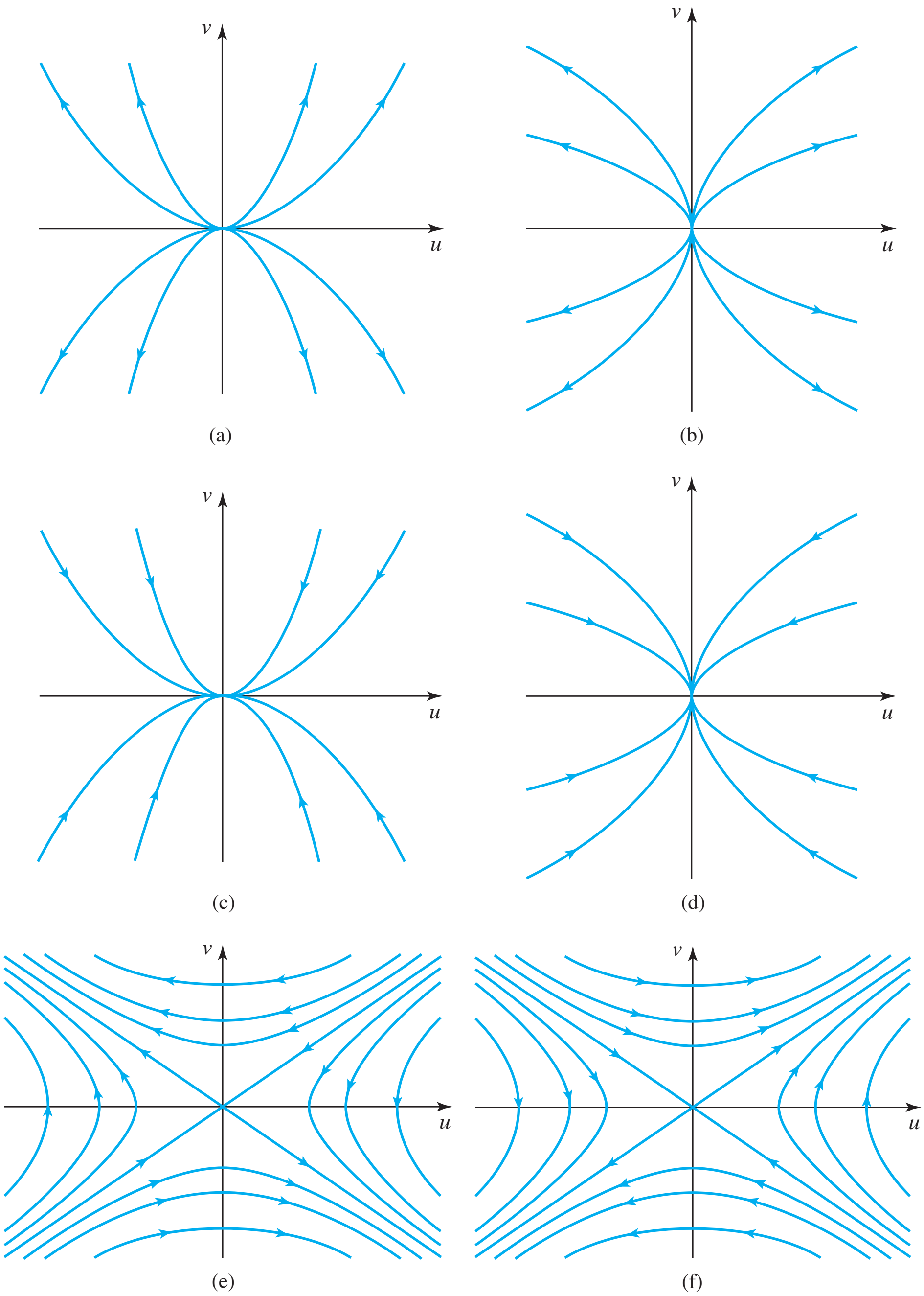


FIGURE 6.14 (a,b) Unstable nodes. (c,d) Stable nodes. (e,f) Saddle points.

types of critical point

the origin is called a **node**. As the eigenvalues are both positive, a point $(u(t), v(t))$ on a trajectory moves *away* from the origin as t increases, so this type of equilibrium point is called an **unstable node**.

(ii) Stable nodes: λ_1 and λ_2 are negative

Examination of the solution in (136) shows that the trajectories must take one of the two forms illustrated in Figs. 6.14c and 6.14d, where the equilibrium point at the origin is again called **node**. This time, as the eigenvalues are both negative, a point $(u(t), v(t))$ on a trajectory will move *toward* the origin as t increases, so in this case the equilibrium point is called a **stable node**.

(iii) Saddle points: λ_1 and λ_2 have opposite signs

Examination of the solution in (136) shows that the trajectories take one of the two forms illustrated in Figs. 6.14e and 6.14f, where the equilibrium point is called a **saddle point**. The eigenvalues are real and have opposite signs, so as t increases a point $(u(t), v(t))$ on a branch of a hyperbola will move toward the origin and then away again, showing that a **saddle point** represents an instability. The two diagonal straight lines that form degenerate hyperbolas are each called a **separatrix** in the phase portrait, because they separate the phase plane into four distinct regions, and a solution in any one of these regions cannot be related to a solution in a different region.

(iv) Degenerate node: Equal eigenvalues $\lambda = \lambda_1 = \lambda_2$

When the discriminant $\Delta = 0$ the eigenvalues coincide, so $\lambda = \lambda_1 = \lambda_2$. In this case the Jacobi matrix $\mathbf{J}$ cannot be diagonalized, but system (132) can always be reduced to the form $d\mathbf{u}/dt = \mathbf{S}\mathbf{u}$, where

$$\mathbf{S} = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix},$$

and this has the general solution

$$u = Ae^{\lambda t} \quad \text{and} \quad v = (At + B)e^{\lambda t}. \tag{137}$$

An examination of solution (137) shows that when $\lambda > 0$, the trajectories are qualitatively similar to the general pattern seen in case (i), corresponding to an equilibrium point that is an **unstable node**. When $\lambda < 0$ the trajectories are qualitatively similar to the general pattern seen in case (ii), corresponding to a **stable node**. Equilibrium points with nodes of this type that arise from coincident eigenvalues are called **degenerate nodes**, so the ones where $\lambda > 0$ are called **unstable degenerate nodes**, and the ones where $\lambda < 0$ are called **stable degenerate nodes**.

Typical patterns of trajectories at unstable degenerate nodes are shown in Figs. 6.15a and 6.15b and at stable degenerate nodes in Figs. 6.15c and 6.15d.

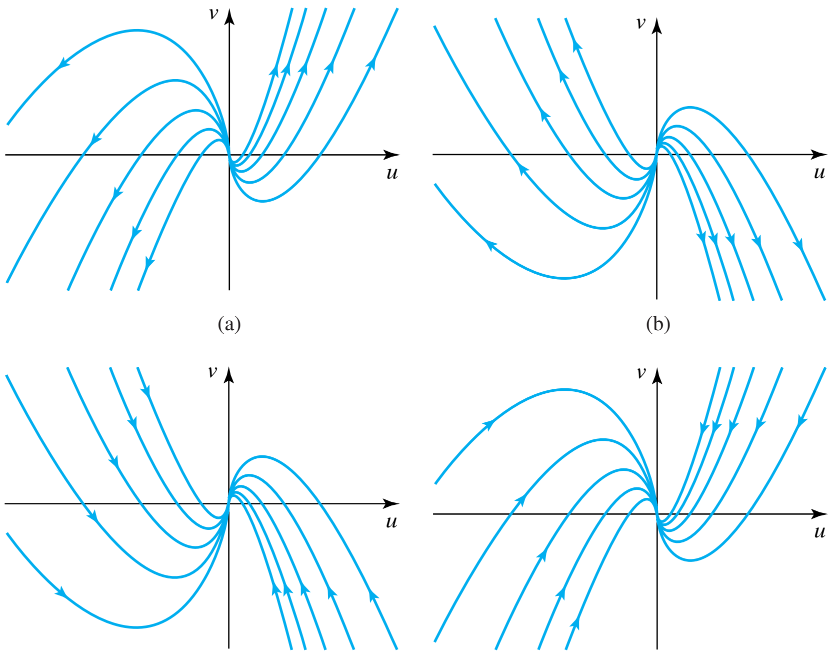


FIGURE 6.15 (a,b) Unstable degenerate nodes. (c,d) Stable degenerate nodes.

(v) Focus or spiral point: Complex conjugate eigenvalues

If the discriminant $\Delta < 0$, the eigenvalues will be the complex conjugates with $\lambda_1 = \xi + i\eta$ and $\lambda_2 = \xi - i\eta$. Diagonalization of $\mathbf{J}$ then produces a system of equations of the form $d\mathbf{u}/dt = \mathbf{C}\mathbf{u}$, where

$$\mathbf{C} = \begin{bmatrix} \xi & \eta \\ -\eta & \xi \end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}.$$

This system is easily shown to have the general solution

$$u = e^{\xi t}(A \sin \eta t - B \cos \eta t) \quad \text{and} \quad v = e^{\xi t}(B \sin \eta t + A \cos \eta t), \tag{138}$$

which defines spiral trajectories about the equilibrium point. In this case the equilibrium point is called a **focus** or a **spiral point**. The direction in which a point $(u(t), v(t))$ along a spiral as t increases is determined by the sign of ξ. When $\xi > 0$ the point moves *away* from the origin as t increases, so the equilibrium point is then called either an **unstable focus** or an **unstable spiral point**. Conversely, when $\xi < 0$, the point moves toward the origin as t increases, so in this case the equilibrium point is called a **stable focus** or a **stable spiral point**. Figure. 6.16a shows an unstable focus and Figure. 6.16b a stable focus. Spirals may evolve in either a clockwise or a counterclockwise direction, and this can be determined by the direction of the vector with components $(dx/dt, dy/dt)$ at any point on the spiral (see Example 6.39).

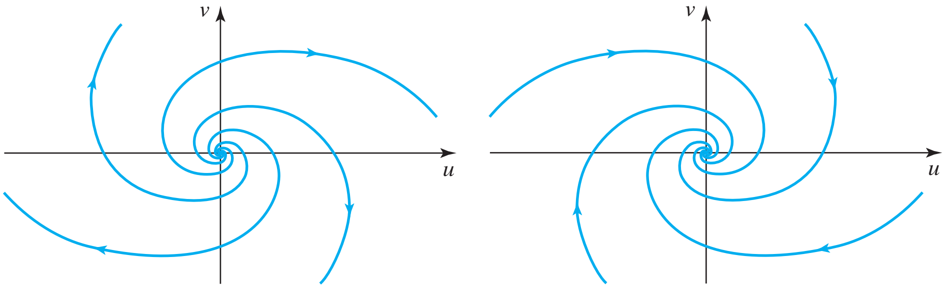


FIGURE 6.16 (a) An unstable focus. (b) A stable focus.

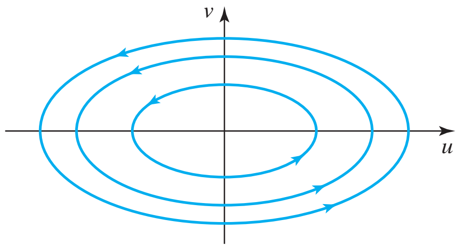


FIGURE 6.17 A center located at the origin.

(vi) Center: Purely imaginary complex conjugate eigenvalues

If in the characteristic equation (135) $\alpha = a - d = 0$ and the discriminant $\Delta < 0$, the eigenvalues will be purely imaginary complex conjugates. Setting $\xi = 0$ in (138) shows that the trajectories become a family of ellipses centered on the origin, as shown in Fig. 6.17. In this case the equilibrium point at the origin is called a **center**, and the corresponding solutions are considered to be *stable* because they remain bounded for all time. It follows from this that the equilibrium point in the linearized predator–prey system is a center.

EXAMPLE 6.37 Locate and identify the nature of the equilibrium point of the system

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -x - 2y,$$

and draw some typical trajectories.

Solution The equilibrium point is located at the origin, and its nature can be identified by examining the eigenvalues of the Jacobi matrix **J** that follows by setting $f(x, y) = -x$ and $g(x, y) = -x - 2y$. We have

$$\mathbf{J} = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix},$$

and this has the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. As the eigenvalues are real, and both are negative, it follows from Case (ii) that the equilibrium point at the origin is a *stable node*. To draw trajectories it is necessary to solve this system, and a routine

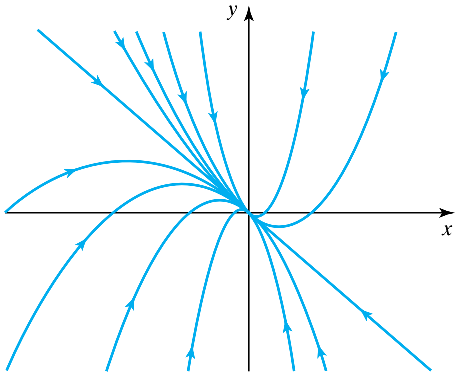


FIGURE 6.18 Trajectories in the neighborhood of the stable node at the origin.

calculation shows that

$$x = -C_1 e^{-t} \text{ and } y = C_1 e^{-t} + C_2 e^{-2t}.$$

Eliminating t, we find that the equation of the trajectories is

$$y = -x + (C_2/C_1^2)x^2.$$

This equation describes a family of parabolas that at the origin are all tangent to the degenerate parabola $y = -x$ that forms a separatrix marking a boundary between phase curves with different properties. Some typical trajectories are shown in Fig. 6.18, where the arrows indicate that the node is stable.

It is important to recognize that as the node is a singularity of the system where dy/dx is indeterminate, a point moving along a trajectory that passes through the node cannot leave it on a different trajectory. ■

EXAMPLE 6.38 Locate and identify the nature of the equilibrium point of the system

$$\frac{dx}{dt} = -x - y - 2, \quad \frac{dy}{dt} = -x + y - 4,$$

and draw some typical trajectories.

Solution The equilibrium point occurs when $-x - y - 2 = 0$ and $-x + y - 4 = 0$, corresponding to $x = -3$, $y = 1$. For convenience we shift the equilibrium point to the origin in the (X, Y) phase plane by making the change of variables $X = x + 3$ and $Y = y - 1$, when the system becomes

$$\frac{dX}{dt} = -X - Y, \quad \frac{dY}{dt} = -X + Y.$$

The nature of the equilibrium point that is now located at the origin in the (X, Y) phase plane can be identified by examining the eigenvalues of the Jacobi matrix

$$\mathbf{J} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix},$$

which are easily seen to be $\lambda_1 = -\sqrt{2}$ and $\lambda_2 = \sqrt{2}$. As the eigenvalues are real, and opposite in sign, it follows from Case (iii) that the equilibrium point at the

origin is a *saddle point*. To draw trajectories it is necessary to solve this system of equations.

After some calculations, the equation of the family of trajectories determined by $dY/dX = (X - Y)/(X + Y)$ is found to be given by

$$Y^2 + 2XY - X^2 = c,$$

where the constant c is determined by the point in the phase plane through which a trajectory is required to pass.

The general equation of a conic is

$$AX^2 + 2BXY + CY^2 + DX + EY + F = 0,$$

and this represents an ellipse if $B^2 - AC < 0$, a parabola if $B^2 - AC = 0$, and a hyperbola if $B^2 - AC > 0$. So comparing the equation of the trajectories with the general form of a conic, we see that $B^2 - AC > 0$, so it describes a family of hyperbolas.

This family of hyperbolas with parameter c is centered on the origin, and solving for Y gives

$$Y = -X + \sqrt{2X^2 + c} \quad \text{and} \quad Y = -X - \sqrt{2X^2 + c},$$

where for any given value of c, each equation represents one pair of hyperbolas.

Some typical hyperbolas are shown in Fig. 6.19, where the upper and lower branches correspond to different values of c in the first equation, and the left and right branches correspond to other values of c in the second equation. The asymptotes, which represent degenerate hyperbolas, are seen by inspection of these equations to be given by $Y = (\sqrt{2} - 1)X$ and $Y = -(\sqrt{2} + 1)X$. Each of these is a *separatrix* in the phase portrait of the system, and a solution in any one of the four regions into which these lines divides the phase plane cannot connect with a solution in any other region.

The simplest way to determine the direction along the upper and lower hyperbolic trajectories as t increases is to find the direction of the vector $(dX/dt, dY/dt)$ on a trajectory. For example, when $X = 0$, we see from the differential equations that the direction of the vector along a trajectory that crosses the Y-axis has the components $(-Y, Y)$. This shows that when $Y > 0$ the vector is directed upward and toward the left, whereas when $Y < 0$ it is directed downward and toward the

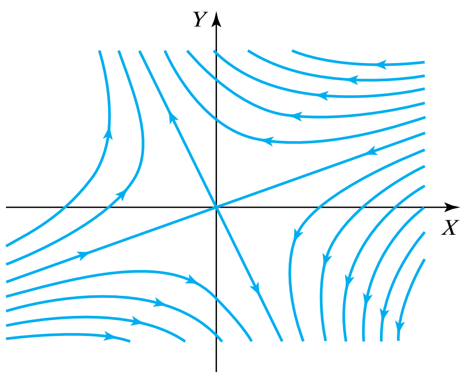


FIGURE 6.19 Trajectories around the saddle point at the origin in the (X, Y) phase plane.

right. The direction of the arrows on the left and right hyperbolic trajectories are determined in similar fashion by finding the direction of the vector $(dX/dt, dY/dt)$ that crosses the X-axis where $Y = 0$.

The pattern of the trajectories around the saddle point in the original coordinate system is obtained by translating the picture in Fig. 6.19 to the point $(-3, 1)$. ■

EXAMPLE 6.39 Locate and identify the equilibrium point of the system

$$\frac{dx}{dt} = -x + 2y + 1, \quad \frac{dy}{dt} = -2x - y + 2,$$

and sketch some trajectories.

Solution The equilibrium point occurs when $-x + 2y + 1 = 0$ and $-2x - y + 2 = 0$, corresponding to $x = 1$ and $y = 0$. For convenience we shift the equilibrium point to the origin in the (X, Y) phase-plane by making the change of variables $X = x - 1$ and $Y = y$, when the system becomes

$$\frac{dX}{dt} = -X + 2Y, \quad \frac{dY}{dt} = -2X - Y.$$

The nature of the equilibrium point that is now located at the origin in the (X, Y) phase plane can be identified by examining the eigenvalues of the Jacobi matrix

$$\mathbf{J} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix},$$

which follows from setting $f(X, Y) = -X + 2Y$ and $g(X, Y) = -2X - Y$.

The eigenvalues are $\lambda_1 = -1 + 2i$ and $\lambda_2 = -1 - 2i$, so as these are complex conjugates with negative real parts, it follows from Case (v) that the equilibrium point at the origin in the (X, Y) phase plane is a *stable focus*. This means that the trajectories spiral into the origin as t increases, so the only question that remains is whether the spiral is clockwise or counterclockwise.

Figure 6.20 shows two possible spirals, where in Fig. 6.20a the direction around the spiral is conterclockwise, while in Fig. 6.20b it is clockwise. Arguing as in Example 6.38, and considering the vector with components $(dX/dt, dY/dt)$ where the spiral crosses the X-axis, by setting $Y = 0$ we find that the vector has components $(-X, -2X)$. As this vector is directed downward and for $x > 0$ to the left, it

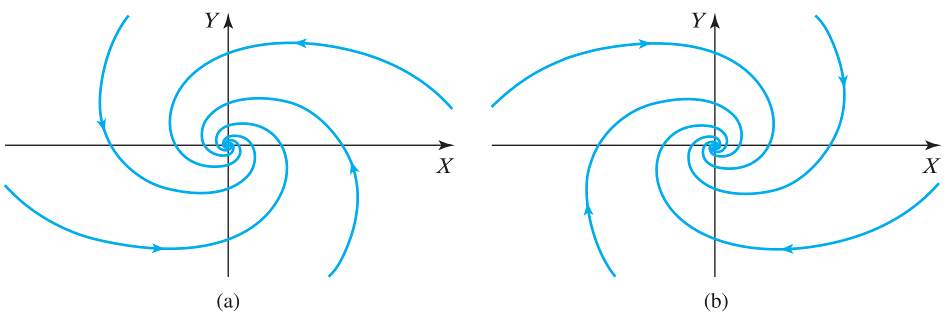


FIGURE 6.20 Two stable foci in the (X, Y) phase plane. (a) Counterclockwise spiral. (b) Clockwise spiral.

follows that the trajectories must spiral clockwise into the origin, so Fig. 6.20b is the only possible phase portrait for this system.

This information is sufficient to enable trajectories to be sketched, but as the general solution of the system is easily found to be

$$X(t) = e^{-t}(c_1 \sin 2t - c_2 \cos 2t), \quad Y(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t),$$

it is not difficult to construct accurate spiral trajectories.

The pattern of trajectories for the original autonomous system is obtained by translating the pattern in Fig. 6.20b to the point $(1, 0)$ in the (x, y) phase plane. ■

If it is only necessary to identify the nature of the equilibrium point at the origin belonging to the linear autonomous system,

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy,$$

identification of critical points

results (i) to (vi) can be summarized as follows:

(a) A **node** if $(a+d)^2 \geq 4(ad - bc) > 0$; stable if $a + d < 0$ and unstable if $a + d > 0$.
(b) A **saddle point** if $ad - bc < 0$.
(c) A **focus** if $(a+d)^2 < 4(ad - bc)$; stable if $a + d < 0$ and unstable if $a + d > 0$.
(d) A **center** if $a + d = 0$ and $ad - bc > 0$.

(vii) Nonlinear autonomous systems

nonlinear autonomous systems

If the nonlinear autonomous system

$$\begin{cases} \dfrac{dx}{dt} = f(x, y) \\ \dfrac{dy}{dt} = g(x, y) \end{cases} \tag{139}$$

has an equilibrium point at (x_0, y_0), the transformation $X = x - x_0$, $Y = y - y_0$ will shift it to the origin in the (X, Y) phase plane. Accordingly, when considering an equilibrium point of system (139), we will always assume that such a translation has been made.

It is plausible to expect that when the nonlinear system in (139) has an equilibrium point at the origin, and in some sense the system is *close* to a linear system, then the nature of the equilibrium point at the origin will be the same in both systems. To make more precise the meaning of the term *close*, we restrict consideration to functions f and g that can be written

$$\begin{aligned} f(x, y) &= ax + by + F(x, y) \\ g(x, y) &= cx + dy + G(x, y), \end{aligned} \tag{140}$$

where $ad - bc \neq 0$ and the nonlinear terms F and G are such that

$$\lim_{x\to 0,\, y\to 0} \frac{F(x, y)}{\sqrt{x^2 + y^2}} = 0 \quad \text{and} \quad \lim_{x\to 0,\, y\to 0} \frac{G(x, y)}{\sqrt{x^2 + y^2}} = 0. \tag{141}$$

This conjecture concerning the relationship between the equilibrium points of a nonlinear and a related linear autonomous system can be shown to be correct, subject only to a single qualification. Specifically, if the linearized system

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned} \tag{142}$$

has a node, a saddle point, or a focus at the origin, then so also has the nonlinear system in (140). The qualification that must be added is that if the equilibrium point at the origin of the linearized system in (142) is a center, then the corresponding nonlinear system in (140) has an equilibrium point at the origin that is *either* a center *or* a focus.

The reason why a center of the linear system (142) may be either a center of a focus of the nonlinear system (140) is not difficult to understand. Conditions (c) and (d) at the end of section (vi) show that the criteria identifying a focus and a center in the linear case are closely related, and it is due to the insensitivity of the linearization process that it fails to distinguish between them when a nonlinear autonomous system is considered. No proof of these statements will be offered here, as this involves methods that do not belong to this first account of autonomous systems. However, a detailed proof of the nature of the relationship between the types of equilibrium points in nonlinear and linearized systems, together with other important results due to Liapunov, Poincaré, and others, can be found in the references at the end of the book.

Nonlinear autonomous systems possess an important property that is not shared by linear systems. This is that in the phase plane a curve Γ may exist, *not* enclosing an equilibrium point, with the property that a trajectory starting from a point either inside or outside Γ is attracted to Γ and spirals into it as t increases. A curve Γ of this type, to which trajectories are attracted, is called a **limit cycle** for the system. Clearly, although a limit cycle represents a stable oscillatory solution, it is *not* one that is asymptotically stable. This statement is essentially the substance of the **Poincaré–Bendixson** theorem, the details of which can be found in the references at the end of the book.

HENRI POINCARÉ (1854–1912)
An outstanding French mathematician who studied in the Ecole Polytechnique in France before proceeding to study in the Ecole Nationale Superieure des Mines in Paris and receiving his doctorate from the University of Paris in 1879. He was appointed to the chair of physical and experimental mechanics at the Sorbonne and later to the chairs of mathematical physics and then the chair of mathematical astronomy. He made fundamental contributions to almost all of mathematics and was probably the last of the mathematical geniuses about whom it could truly be said that he knew all that was then known about mathematics.

It was proved separately by Bendixson that if in system (139) the functions f and g have continuous partial derivatives for all x and y, and $f_x + f_y$ is either positive or negative in some region Ω of the phase plane, then the system has no limit cycle in Ω. Although the proof of this result is not difficult, it will not be given here. The result is useful for establishing the nonexistence of limit cycles in given regions of the phase plane.

A theorem that gives sufficient, though not necessary, conditions for the existence of a limit cycle for a special type of autonomous system is Liénard's theorem. The theorem is now stated without proof.

THEOREM 6.9

conditions identifying a limit cycle: Liénard's theorem

Liénard's theorem Write the linear equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

as the first order Liénard system

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -g(x) - f(x)y.$$

Let $f(x)$ and $g(x)$ satisfy the following conditions:

(i) $f(x)$ and $g(x)$ are continuous functions with continuous first derivatives for all x.

(ii) $g(x)$ is an odd function that is positive for $x > 0$ and $f(x)$ is an even function.

(iii) the function $F(x) = \int_0^x f(\xi)d\xi$, which is an odd function, has precisely one positive root at $x = \alpha$, with $F(x) < 0$ for $0 < x < \alpha$, $F(x) > 0$ and nondecreasing for $x > \alpha$, and $F(x) \to \infty$ as $x \to \infty$.

Then the Liénard system possesses a unique closed curve Γ enclosing the origin in the phase plane, with the property that every trajectory spirals toward Γ as $t \to \infty$. ■

van der Pol equation and phase portraits

An application of this theorem will be made later to the **van der Pol equation**

$$\frac{d^2x}{dt^2} + \varepsilon(x^2 - 1)\frac{dx}{dt} + x = 0, \tag{143}$$

which provides a classical example of a limit cycle. The equation itself was derived in the 1920s by Balthazar van der Pol when studying self-sustained oscillations in vacuum tubes, and it was his work that prompted Liénard to study corresponding problems in nonlinear mechanics.

The task of finding the complete phase portrait of a nonlinear autonomous system, usually called the **global phase portrait**, can be difficult. This is because nonlinear systems may have more than one equilibrium point, and while linearization techniques provide information in a neighborhood of each of these points (with the exception of centers), they provide very little information about the general phase portrait or any separatrix that may occur, and no information at all about the existence of a limit cycle, though Liénard's theorem helps in the linear case.

The Predator–Prey Problem

more on the predator–prey problem

The predator–prey equations have been shown to have a single physically meaningful equilibrium point at $(c/d, a/b)$ in the phase plane, where the linearized form of the equations has a center with elliptical trajectories surrounding it. In view of the fact that when the linearized form of a nonlinear system identifies an equilibrium point as a center, the associated nonlinear system may have either a center or a focus, a more careful examination is necessary in the predator–prey case before it is possible to state with certainty that $(c/d, a/b)$ is a center and that cyclic variations in the populations take place. In more advanced accounts of nonlinear autonomous systems, theorems exist that can resolve this ambiguity, but here we will make use of a simple device that in this and other straightforward cases will suffice to distinguish between the two possibilities.

The idea is simple, and it involves asking how many times a trajectory will intersect a straight line drawn through the equilibrium point at $(c/d, a/b)$. If the equilibrium point is a center, a trajectory can only intersect this line twice, but if it is a focus (a spiral point) it will intersect it infinitely many times.

Dividing the second of the predator–prey equations in (122) by the first equation, rearranging terms, and integrating gives

$$\int \frac{(a-by)}{y}dy = \int \frac{(xd-c)}{x}dx,$$

and so

$$a \ln y + c \ln x - by - xd = k,$$

where k is an integration constant. To proceed further, we consider a typical case where $a = 1$, $b = 1$, $c = 2$, and $d = 1$, when the predator–prey system will have an equilibrium point at (2, 1) in the phase plane, and the equation determining the trajectories becomes

$$\ln y + 2 \ln x - y - x = k.$$

Let us now select a convenient trajectory through any point in the first quadrant that does not coincide with the equilibrium point. It is convenient to choose the point (1, 1), when it follows from the above equation that $k = -2$, so the equation of the trajectory through this point becomes

$$\ln y + 2 \ln x - x = -3.$$

We may choose any test line through the equilibrium point, but it is simplest to choose the line $y = 1$ that passes through the equilibrium point of the system at (2, 1) in the phase plane. Setting $y = 1$ in the preceding equation reduces it to

$$2 \ln x - x = -3,$$

so if this equation has only two real roots the equilibrium point will be a center, but if it has infinitely many it will be a focus. Graphing $y = 2 \ln x - x$ and $y = -3$ to determine where they intersect, we find that only two intersections occur, with one at $x \approx 0.25$ and the other at $x \approx 6.85$. This shows that in this model of the predator–prey system the equilibrium point at (2, 1) must be a center.

A similar argument applies to any other choice of nonnegative coefficients a, b, c, and d. This demonstrates that the equilibrium point of the predator–prey system located at (2, 1) in the first quadrant of the phase plane is, indeed, a center.

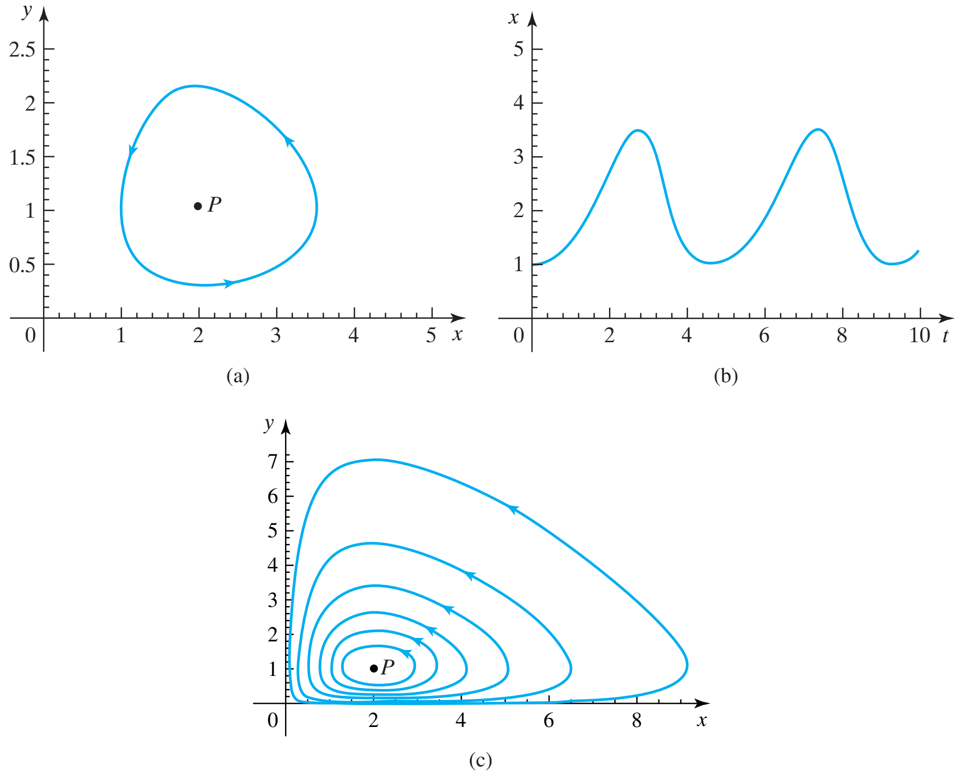


FIGURE 6.21 (a) The phase plane for the system through the point (1, 1) with an equilibrium point at (2, 1). (b) The variation of $x(t)$ showing the cycle time to be approximately 4.7 time units. (c) A general family of trajectories, each with the same equilibrium point.

Negative rabbit and fox populations have no physical significance, so no attention need be paid to the saddle point located at the origin of the phase plane, but notice that each axis is a separatrix belonging to the saddle point. Accordingly, the computer-generated phase portrait in the first quadrant is shown in Fig. 6.21a, with $a = 1$, $b = 1$, $c = 2$, and $d = 1$, the rabbit population along the horizontal axis, and the fox population along the vertical axis. The equilibrium point is shown as P. To find the period of this cycle of events, it is sufficient to find the period of either $x(t)$ or $y(t)$. The variation of $x(t)$ is shown in Fig. 6.21b with t along the horizontal axis and x along the vertical axis, from which the period is seen to be approximately $T \approx 4.7$ time units. Figure 6.21c shows a general family of trajectories for this system, each with a different period.

The Undamped and Damped Simple Pendulum

study of the undamped and damped pendulum

The geometry of the simple pendulum is illustrated in Fig. 6.22, where a mass m is attached to the end of a light rigid rod of length l that is pivoted at the end opposite to the mass and allowed to oscillate under gravity. The equation of motion, when damping proportional to $d\theta/dt$ is present, can be written

$$ml^2\frac{d^2\theta}{dt^2} + 2mlk\frac{d\theta}{dt} + mgl\sin\theta = 0,$$

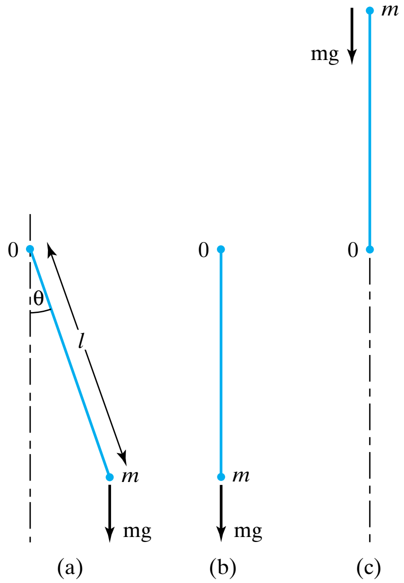


FIGURE 6.22 (a) Small oscillations. (b) Stable equilibrium. (c) Inverted pendulum—unstable equilibrium.

where $k > 0$ is a constant. Here, to simplify the associated characteristic equation, the constant of proportionality for wind resistance has been set equal to $2mlk$. This is equivalent to setting $\mu = 2mk/l$ in the equation of motion for a damped pendulum derived at the start of Section 6.1.

The undamped pendulum

Let us start by considering the undamped case $k = 0$. Introducing the new variable $x = d\theta/dt$, we see the nonlinear autonomous system determining the motion to be

$$\frac{dx}{dt} = -\left(\frac{g}{l}\right)\sin\theta \quad \text{and} \quad \frac{d\theta}{dt} = x,$$

with equilibrium points on the θ-axis where $\sin\theta = 0$. This shows there are infinitely many equilibrium points along the θ-axis at $\theta = \pm n\pi$, for $n = 0, 1, \ldots$. Accordingly, because of the periodicity of $\sin\theta$, only the interval $-\pi \leq \theta \leq \pi$ need be considered.

If we write $\sin\theta = \theta + (\sin\theta - \theta)$, the system becomes

$$\begin{aligned}\frac{dx}{dt} &= -\left(\frac{g}{l}\right)\theta - \left(\frac{g}{l}\right)(\sin\theta - \theta)\\ \frac{d\theta}{dt} &= x.\end{aligned}$$

The nonlinear term $(g/l)\,(\sin\theta - \theta)$ satisfies the condition in (141), so when the equilibrium point at the origin is considered, the Jacobi matrix becomes

$$\mathbf{J} = \begin{bmatrix} 0 & -g/l \\ 1 & 0 \end{bmatrix}.$$

This has the purely imaginary eigenvalues $\lambda_1 = -i\sqrt{(g/l)}$ and $\lambda_2 = i\sqrt{(g/l)}$, so the equilibrium point of the linearized system located at the origin is a center. An argument similar to the one used with the predator–prey equations can be used to show that any trajectory starting at a point on the line $\theta = 0$ in the interval $-\pi < \theta < \pi$ will intersect the x-axis twice, so the equilibrium point of the nonlinear system is also a center. This confirms the expected result that the pendulum will perform periodic oscillations.

Next we must consider the equilibrium point at $(\pi, 0)$, and to do this we shift the origin of the system to this point by setting $u = \theta - \pi$. This causes the equation $dx/dt = -(g/l)\sin\theta$ to become $dx/dt = (g/l)\sin u$, so the system can now be written

$$\frac{du}{dt} = x$$
$$\frac{dx}{dt} = \left(\frac{g}{l}\right)u + \left(\frac{g}{l}\right)(\sin u - u).$$

The nonlinear term again satisfies the conditions in (141), so the nature of this equilibrium point is determined by the eigenvalues of the Jacobi matrix $\mathbf{J}$, which now becomes

$$\mathbf{J} = \begin{bmatrix} 0 & g/l \\ 1 & 0 \end{bmatrix}.$$

This has the real eigenvalues $\lambda_1 = -\sqrt{(g/l)}$ and $\lambda_2 = \sqrt{(g/l)}$, so as these are of opposite sign the equilibrium point at $(\pi, 0)$ is seen to be a saddle point. An analogous argument shows that the equilibrium point at $(-\pi, 0)$ is also a saddle point, so the nonlinear system also has saddle points at $(\pm\pi, 0)$.

A repetition of these arguments shows the equilibrium points at $(\pm 2n\pi, 0)$ all to be centers, and the equilibrium points at $((2n+1)\pi, 0)$ all to be saddle points. A computer plot of some typical trajectories is shown in Fig. 6.23a.

An examination of Fig. 6.23a explains the significance of these centers and saddle points. As the angular displacement of the pendulum is indeterminate up to a multiple of 2π, each center represents the stable nonlinear oscillations that occur in Fig. 6.22a when the pendulum never becomes inverted. Similarly, each saddle point represents the unstable position of the inverted pendulum shown in Fig. 6.22c. As the oscillations are nonlinear, each different closed curve about a center represents a nonlinear oscillation with a different period. Each dashed curve is a separatrix forming a boundary between phase curves with different properties.

An important and useful result is obtained by writing

$$\frac{d^2\theta}{dt^2} = \frac{dx}{dt} = \frac{d\theta}{dt}\frac{dx}{d\theta} = x\frac{dx}{d\theta}.$$

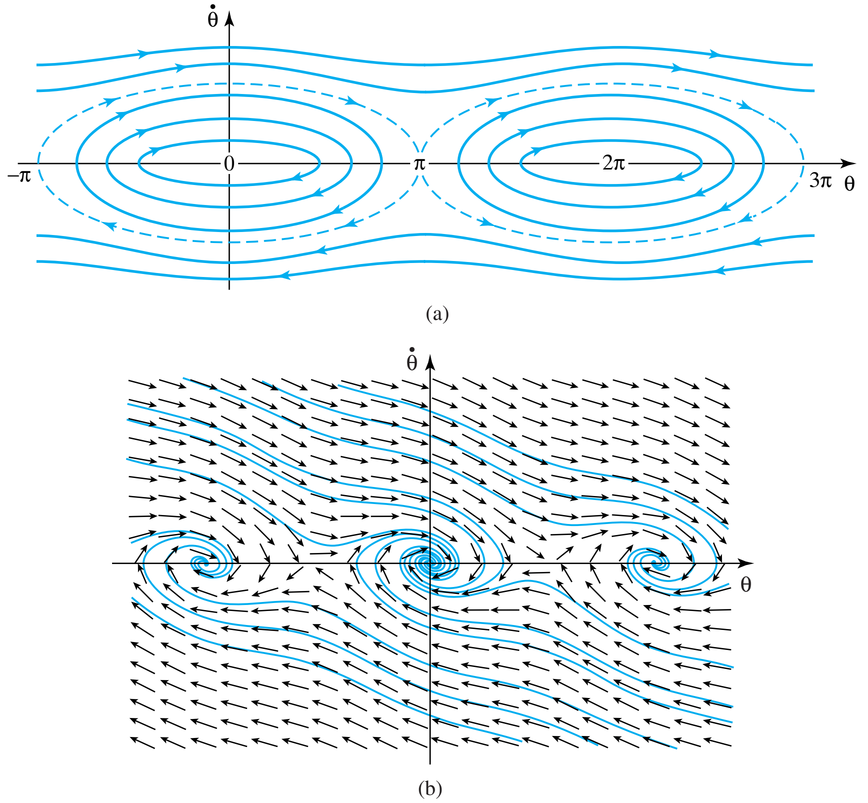


FIGURE 6.23 (a) The phase portrait for the undamped pendulum. (b) The phase portrait for the damped pendulum.

Using this result the equation of motion becomes

$$ml^2 x\frac{dx}{d\theta} + mgl\sin\theta = 0,$$

so after integration we have

$$\frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 - mgl\cos\theta = C,$$

where C is an integration constant.

This first integral of the equation of motion expresses the *conservation of energy* in the system, which is possible because when $k = 0$ there is no dissipation of energy due to friction.

The damped pendulum

When damping occurs ($k > 0$), the nonlinear autonomous system governing the oscillations of the pendulum becomes

$$\frac{d\theta}{dt} = x \quad \text{and} \quad \frac{dx}{dt} = \frac{-2kx}{l} - \left(\frac{g}{l}\right)\sin\theta.$$

Considering the equilibrium point that again occurs at the origin, we write the system as

$$\frac{d\theta}{dt} = x$$
$$\frac{dx}{dt} = \frac{-2kx}{l} - \left(\frac{g}{l}\right)\theta - \left(\frac{g}{l}\right)(\sin\theta - \theta).$$

Then, proceeding as before, we see that the nature of the equilibrium point at the origin is determined by the eigenvalues of the Jacobi matrix

$$\mathbf{J} = \begin{bmatrix} -2k/l & -g/l \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation of $\mathbf{J}$ is

$$\lambda^2 + (2k/l)\lambda + g/l = 0,$$

so as $\lambda = -k/l \pm \sqrt{k^2 - lg}/l$, and as $k > 0$, the eigenvalues are real and negative when $k > g/l$, corresponding to overdamped oscillations. When $(k/l)^2 < g/l$ the eigenvalues are complex conjugates with negative real parts, corresponding to the asymptotically stable oscillatory case. So, when friction is present, the equilibrium point at the origin is seen to be an asymptotically stable focus. In time, friction will cause the oscillations to decay to zero, causing the pendulum to come to rest in the positions shown in Fig. 6.23b.

EXAMPLE 6.40 Locate and classify the equilibrium points of the nonlinear autonomous system

$$\frac{dx}{dt} = 4 - x^2 - 4y^2 \quad \text{and} \quad \frac{dy}{dt} = xy.$$

Solution The equilibrium points occur when $4 - x^2 - 4y^2 = 0$ and $xy = 0$, so the points are located at $(0, -1)$, $(0, 1)$, $(2, 0)$, and $(-2, 0)$. Let us consider the equilibrium point at $(0, 1)$ and shift the origin to this point by setting $Y = y - 1$ and $X = x$. The system now becomes

$$\frac{dX}{dt} = -8Y - X^2 - 4Y^2 \quad \text{and} \quad \frac{dY}{dt} = X + XY.$$

Setting $X = r\cos\theta$, $Y = r\sin\theta$, we easily see that conditions (141) are satisfied, so the nature of the equilibrium point at $(0, 1)$ will be determined by the eigenvalues of the Jacobi matrix

$$\mathbf{J} = \begin{bmatrix} 0 & -8 \\ 1 & 0 \end{bmatrix}.$$

These satisfy the characteristic equation $\lambda^2 + 8 = 0$, so as they are purely imaginary, the equilibrium point of the linearized system that is located at $(0, 1)$ must be a center, and arguments similar to those used with the pendulum problem confirm that the nonlinear system also has a center at $(0, 1)$.

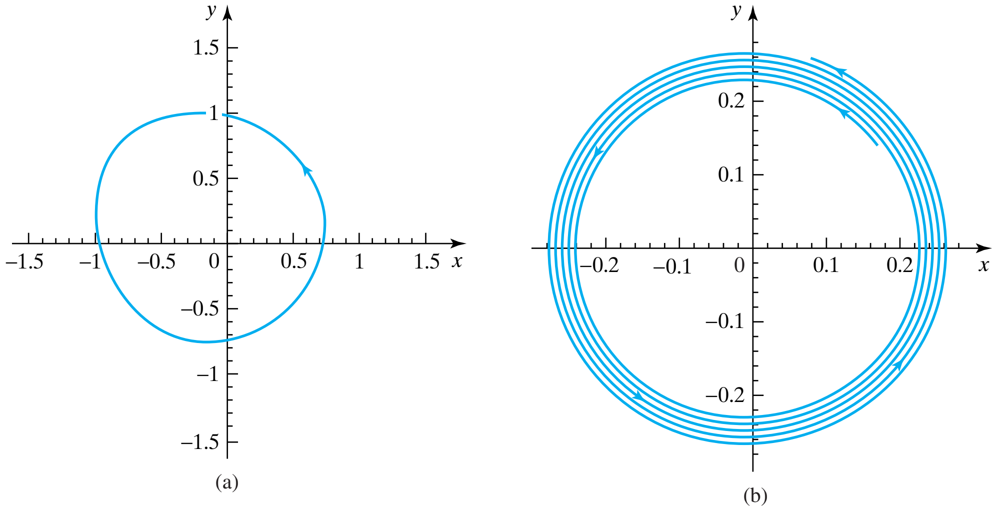


FIGURE 6.24 (a) The origin is a center. (b) The origin is an unstable focus.

It is left as an exercise to use similar arguments to show that the equilibrium point at $(0, -1)$ is also a center and the equilibrium points at $(-2, 0)$ and $(2, 0)$ are saddle points. ■

The inability of a linearized system to reflect the difference between a center and a focus in the nonlinear system from which it is derived is best illustrated by means of computer-generated phase portraits. The following two systems only differ in the power of x associated with dx/dt, and each has the same linearized form that indicates the existence of a center at the origin of the phase plane:

$$\text{(i)} \qquad \frac{dx}{dt} = -4y + x^2 \quad \text{and} \quad \frac{dy}{dt} = 4x + y^2$$

and

$$\text{(ii)} \qquad \frac{dx}{dt} = -4y + x^3 \quad \text{and} \quad \frac{dy}{dt} = 4x + y^2.$$

However, the nonlinear phase portrait of system (i) in Fig. 6.24a shows that the system does, indeed, have a center located at the origin in the phase plane, but the nonlinear phase portrait of system (ii) in Fig. 6.24b shows that the system has an unstable focus at the origin.

A typical example of a limit cycle is provided by the van der Pol equation

$$\frac{d^2x}{dt^2} + \varepsilon(x^2 - 1)\frac{dx}{dt} + x = 0.$$

If we set $f(x) = \varepsilon(x^2 - 1)$ and $g(x) = x$ in Liénard's theorem, it is easily seen that the conditions of the theorem are satisfied provided $F(x) = \int_0^x \varepsilon(\xi^2 - 1)d\xi$ has precisely one positive root $x = \alpha$ with $F(x) < 0$ for $0 < x < \alpha$, and $F(x)$ is such that it is positive and nondecreasing for $x > \alpha$ with $F(x) \to \infty$ as $x \to \infty$. This

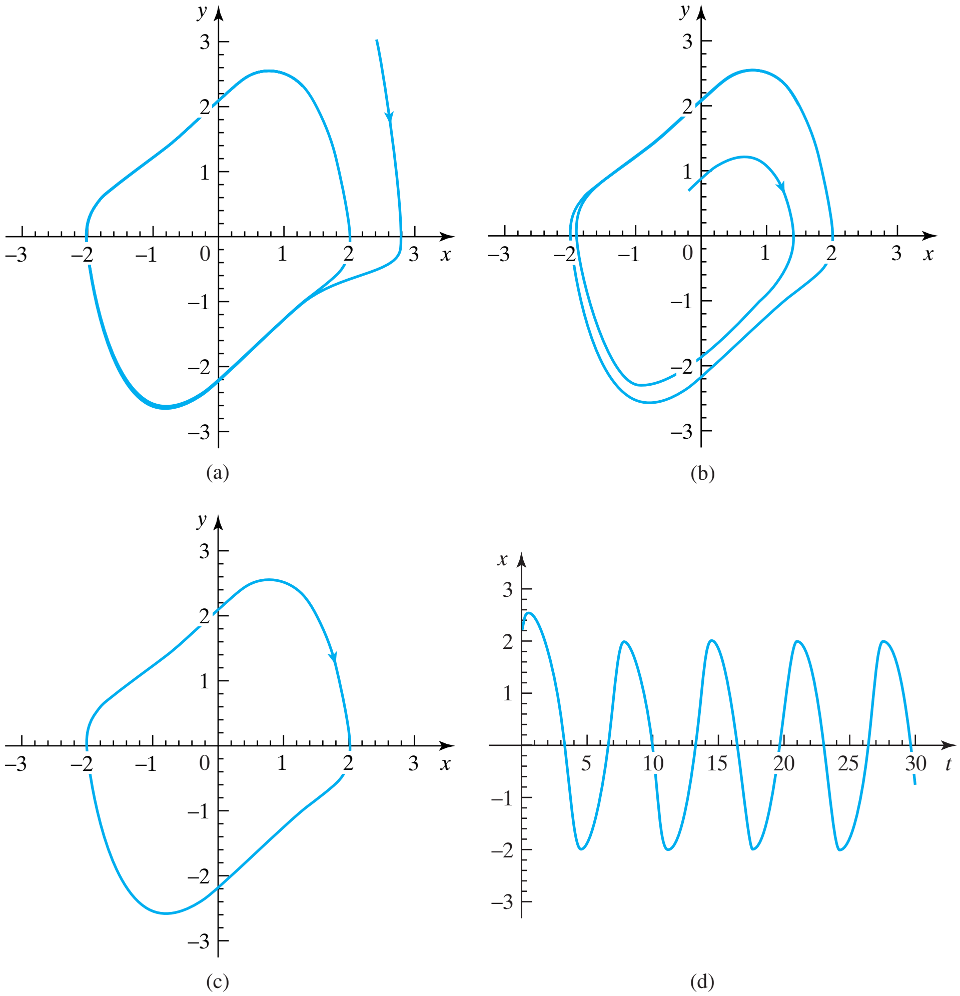


FIGURE 6.25 Phase portraits for the van der Pol equation with $\varepsilon = 0.9$ and the variation of $x(t)$ with t. (a) A trajectory starting outside the limit circle. (b) A trajectory starting inside the limit cycle. (c) The limit cycle. (d) The periodicity of $x(t)$ as a function of t.

is seen to be the case, because $F(x) = \frac{1}{3}\varepsilon(x^3 - 3x)$, so the theorem ensures the existence of a limit cycle for the van der Pol equation provided $\varepsilon > 0$.

Figure 6.25a shows a computer-generated phase portrait for the van der Pol equation with $\epsilon = 0.9$, where the trajectory starting from an initial point at $t = 0$ outside the limit cycle (the parallelogram-shaped closed curve) is attracted *inward* toward the limit cycle. Figure 6.25b shows the corresponding situation when the initial point lies inside the limit cycle, where here the trajectory is attracted *outward* toward the limit cycle. Figure 6.25c shows the limit cycle itself. A plot of $x(t)$ against t is shown in Fig. 6.25d, from which the solution is seen to become periodic, with a period of approximately 6.5 time units, after the time $t = 5$.

More examples of the phase plane are to be found in references [3.3] to [3.5], whereas a more extensive and advanced account is to be found in references [3.1], [3.2], and [3.13].

Summary

An autonomous system involving the variables $x(t)$ and $y(t)$, where the parameter t is usually the time, are systems of the form $dx/dt = f(x, y)$ and $dy/dt = g(x, y)$, where the dependence of the f and g on t is implicit. Critical points of such systems were defined and the concept of a trajectory, or path, was introduced leading to the notion of a phase portrait. Stability, instability, and asymptotic stability were defined, and the classical predator–prey problem was used to illustrate ideas. Linearization of the functions f and g led to the identification of different types of critical points for linear autonomous systems. These ideas were extended to nonlinear autonomous systems where it was possible for trajectories to spiral in or out until they entered a closed loop called a limit cycle, where the solution became periodic, though nonlinear. These ideas were illustrated by application to the full nonlinear predator–prey problem, the pendulum problem, and the van der Pol equation.

EXERCISES 6.12

In Exercises 1 through 6, locate and identify the nature of the equilibrium point and sketch the pattern of the trajectories.

1. $dx/dt = y, \quad dy/dt = x.$

2. $dx/dt = x + 2, \quad dy/dt = -x + 2y - 8.$

3. $dx/dt = x - 2y, \quad dy/dt = 4x - 3y.$

4. $dx/dt = x - y, \quad dy/dt = 2x - y.$

5. $dx/dt = x + 3y - 4, \quad dy/dt = -6x - 5y + 22.$

6. $dx/dt = 2y - x, \quad dy/dt = 3x + 6.$

In Exercises 7 through 9 locate the equilibrium points of the given nonlinear autonomous system and, where possible, use linearization to identify their nature.

7. $dx/dt = x^2 - y^2 - 4, \quad dy/dt = y.$

8. $dx/dt = 2 + y - x^2, \quad dy/dt = x^2 - xy.$

9. $dx/dt = x + y + y^2, \quad dy/dt = 2x + y.$

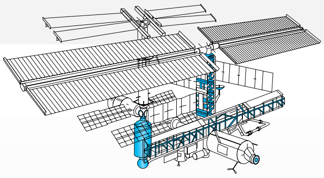
10. Locate and identify the equilibrium points of

$$dx/dt = -x + xy, \quad dy/dt = 3y - 2xy + x.$$

11. Show that the only equilibrium point of the van der Pol equation

$$\frac{d^2x}{dt^2} + \varepsilon(x^2 - 1)\frac{dx}{dt} + x = 0$$

is located at the origin. By linearizing the equation about the origin, find conditions that must be imposed on ε in order that (a) the equilibrium point be an unstable spiral, (b) that it be an unstable node, and (c) that it be a center. Relate your results to the phase portraits in Fig. 6.25.



CHAPTER 6
TECHNOLOGY PROJECTS

The purpose of the first two projects is to use a computer algebra phase portrait package to construct the phase portraits for linear and nonlinear systems, and to examine the nature of the limit cycles in the van der Pol equation for different choices of the parameter ε and the initial conditions.

Project 1

Phase Portraits

Use a computer phase portrait package to construct the phase portraits for the following systems about the origin:

(a) $\dfrac{dx}{dt} = 2x^2 - 3y, \quad \dfrac{dy}{dt} = x + 2y.$

(b) $\dfrac{dx}{dt} = x + 2y, \quad \dfrac{dy}{dt} = x - 3y.$

(c) $\dfrac{dx}{dt} = 2x - 3y, \quad \dfrac{dy}{dt} = -x + 2y.$

(d) $\dfrac{dx}{dt} = 2x - 4y, \quad \dfrac{dy}{dt} = 4x - 2y.$

(e) $\dfrac{dx}{dt} = x + 3y^2, \quad \dfrac{dy}{dt} = x + 2y.$

Project 2

The Limit Cycle of the van der Pol Equation

Use a computer algebra phase portrait package to construct integral curves for the van der Pol equation

$$x'' + \varepsilon(x^2 - 1)x' + x = 0$$

for $\varepsilon = 0.5$, 1.0, and 1.5, starting trajectories from points inside and outside the limit cycle shown in Fig. 6.25.

Project 3

Period of Oscillation of a Nonlinear Pendulum

The nonlinear equation of motion of a simple pendulum when the mass of the pendulum rod is neglected is

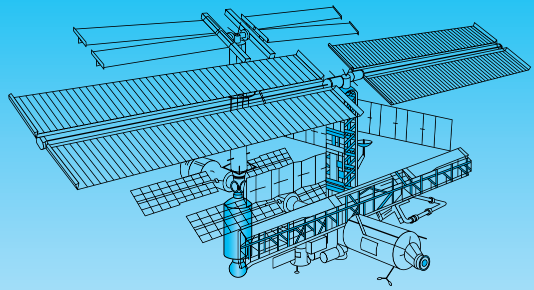
$$m\phi'' + (mg/l)\sin\phi = 0,$$

where a prime denotes differentiation with respect to the time t, m is the mass of the pendulum bob, g is the acceleration due to gravity, l is the length of the pendulum, and ϕ is the angle of deflection of the pendulum from the vertical. When the maximum angle of deflection of the pendulum from the vertical is θ, the period of oscillation T is given by the complete elliptic integral

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2} \frac{du}{(1 - \sin^2(u)\sin^2(\frac{1}{2}\theta))^{1/2}}. \qquad \text{(I)}$$

1. Use the numerical integration facility of MAPLE to find $(T/4)\sqrt{(g/l)}$ for some specific θ.
2. Expand the integrand of (I) as a Maclaurin series in u and integrate term by term to find a series representation for $(T/4)\sqrt{(g/l)}$ in terms of powers of $\sin\theta$.
3. Set $\theta = 2\pi/5$ and approximate the result in Part 2 by taking the first N terms, with $N = 2^m$ and $m = 1, 2, \ldots$. By repeatedly doubling N and comparing the estimate of $(T/4)\sqrt{(g/l)}$ with the result obtained in Part 1, find how many terms must be used in the approximation if the result is to agree to four decimal places.

CHAPTER 7



The Laplace Transform

Many problems in engineering and physics can be described in terms of the evolution of solutions of linear differential equations subject to initial conditions. An important group of these problems involves constant coefficient differential equations, and equations like these can be solved very easily by using the Laplace transform.

The Laplace transform is an integral transform that changes a real variable function $f(t)$ into a function $F(s)$ of a variable s through

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

where in general s is a complex variable.

The importance of the Laplace transform in the study of initial value problems for linear constant coefficient differential equations is that it replaces the operation of integrating a differential equation in $f(t)$ by much simpler algebraic operations involving $F(s)$. Unlike previous methods, where first a general solution is found, and then the constants in the complementary function are chosen to match the initial conditions, when the Laplace transform method is used the initial conditions are incorporated from the start. The task of finding the function $f(t)$ from its Laplace transform $F(s)$ is called inverting the transform, and when working with constant coefficient equations we can accomplish this by appeal to tables of Laplace transform pairs—that is, to a table listing a function $f(t)$ and its corresponding Laplace transform $F(s)$.

The fundamental ideas underlying the Laplace transform are derived, along with its operational properties, which are illustrated by examples. Initial value problems for ordinary differential equations are solved by the Laplace transform, which is then applied to systems of equations and to certain variable coefficient equations. The chapter concludes with applications of the Laplace transform to a variety of problems, the last of which is the heat equation.

7.1 Laplace Transform: Fundamental Ideas

Let the real function $f(t)$ be defined for $a \le t \le b$, and let the function $K(t, s)$ of the variables t and s be defined for $a \le t \le b$ and some s. When it exists, the